# MATHEMATICS FOR MECHANICS

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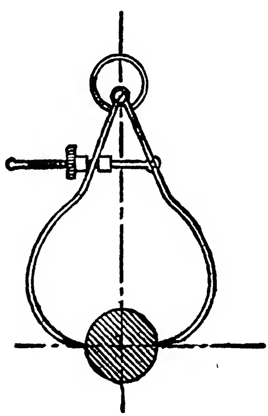
By

WILLIAM L. SCHAAF, Ph.D.

*Assistant Professor of Education*
*Brooklyn College*



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# MATHEMATICS FOR MECHANICS

By William L. Schaaf, Ph.D.
*Assistant Professor of Education, Brooklyn College*

HERE AT LAST is a "math" book you can *really use* in your shop work. It will help you to solve hundreds of practical everyday problems—easily, quickly.

*Mathematics for Mechanics* is precisely what the title says. It gives you the knowledge you want—shop trigonometry, practical geometry, elements of algebra, and fundamental arithmetic. Simple, easy-to-understand instructions and hundreds of examples and illustrations show you plainly how to figure out any problem you need to, and *know* you are *right*.

If you know your "math" you are in line for bigger pay. That's the way to get singled out for promotion; it's mastery of mathematics that will make you able to supervise others.

The author of this book, Professor William L. Schaaf, of Brooklyn College, has had years of experience in teaching the kind of mathematics that mechanics need in their daily work. He knows just how to explain and illustrate every point and problem. As a result you can understand anything in

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# INTRODUCTION

We are living in an age in which mechanical arts and industrial production play a central role. The complexity of modern industry, as well as increased specialization in the various trades, have led to more exacting demands on the technical knowledge needed by the skilled mechanic. Prominent among these needs is an effective, working knowledge of mathematics. Indeed, it is no exaggeration to say that a mastery of basic mathematics is indispensable. The machine shop worker and the mechanic in almost every trade are constantly called upon to use certain fundamental mathematical facts and procedures. This is true not only of their daily activities on the job, but even more so when promotion and advancement are considered. The successful foreman is the man who, among other qualifications, can use his mathematics with confidence.

A number of practical, introductory books on mathematics have been written for laymen and for businessmen; but books on mathematics for *workmen* are less common. To be sure, many textbooks for machine shop students are available, but they generally include considerable specialized and advanced, difficult material. This book is precisely what the title says it is—a practical, introductory treatment of the mathematics essential for mechanics and workmen. In the shop and in the factory the worker must often perform tasks involving (1) computation, (2) measurement, (3) layout work, (4) interpretation of a blueprint, and (5) the use of tables and graphs. To carry out these activities, specific mathematical concepts and skills are required. A careful survey of the most important trade activities reveals which phases of arithmetic, algebra, geometry, and trigonometry will furnish the necessary mathematical facts and methods. It is these very phases which are encompassed here, constituting what might properly be called the "basic essentials" of the mathematics for workmen.

Experience shows that for the practical workman the most important parts of arithmetic include: the ability to use fractions and decimals with

accuracy and facility; an understanding of ratio and proportion, and of percentage; familiarity with weights and measures; and a knowledge of the theory and use of precision measuring instruments.

As for the most useful skills of elementary algebra, these include the following: the use of symbols and formulas; positive and negative numbers; exponential notation; finding square roots; the solution of simple equations; the graphic representation of formulas and equations; the use of logarithms; and familiarity with the slide rule.

The contributions of geometry are also important. These include: the fundamental properties of common plane and solid geometric figures; the measurement of angles; the execution of certain basic geometric constructions; ability to measure and compute perimeters and areas of plane figures; ability to compute the surfaces and cubical contents and volumes of solid figures; the use of scale drawings; and an understanding of similar figures, both plane and solid.

Certain features of elementary plane trigonometry are most helpful, such as: the meaning of the sine, cosine and tangent of an angle; the use of trigonometric tables; the solution of right triangles; the trigonometric relations needed for the solution of oblique triangles; and methods of handling projections and special figures such as regular polygons.

In writing a book of this kind, two questions invariably arise: (1) how much shop information and trade knowledge should be included to illustrate the mathematical ideas and procedures, and (2) to what extent should the treatment be theoretical and explanatory, rather than merely a collection of empirical formulas and rule-of-thumb methods. Both of these problems, it is believed, have been happily solved. As for the first question, sufficient illustrative material has been drawn from the fields of physics, mechanics, drafting, machine shop, woodworking, metal trades, electrical trades and general engineering to demonstrate adequately the utility and actual application of the mathematical principles and processes, without, however, becoming a "handbook" or "textbook" in some particular trade. As for the second problem, *general methods* have been stressed in preference to specific methods or special tricks. Thus, for example, in the discussion of formulas and equations, skill in the transformation of formulas and the solution of equations in general has been emphasized. Or again, in the treatment of square root extraction, several methods and explanations are given, even though appropriate tables are furnished and recommended.

In conclusion, the author wishes to acknowledge his indebtedness to the following firms for permission to use published material: International Business Machine Corporation; Keufel and Esser Co.; McGraw-Hill Book Company; L. S. Starrett Company; and D. Van Nostrand Company.

W. L. S.

# PART I

# BASIC MATHEMATICS

# CHAPTER I

# FUNDAMENTALS OF ARITHMETIC

*William L. Schaaf*

## 1. WEIGHTS AND MEASURES

**Importance of Measurement.** In the mechanized civilization in which we live the vital importance of measurement cannot be overemphasized. Careful measurements are indispensable in business, in almost all trades, in manufacturing and industry, to say nothing of science, technology and engineering. A famous scientist, Lord Kelvin, once said that not until we can measure something can we really understand it. The same is true of design and production; finished products simply could not be made without the use of measurements. One has only to think of the many parts of an auto engine, the delicate mechanism of a watch, the sensitive shutter of a camera, or the precision of a lathe, to realize how utterly dependent modern man is upon the art of measurement. Indeed, the history of the development of units of measure reflects the march of civilization in more ways than one.

**Nature of Measurements.** To understand fully the nature and use of measurements, two basic principles must be borne in mind:

(1) *All measurements are approximate.*
(2) *All units of measure are arbitrary.*

There is no such thing as an absolute or perfect measurement. An object can be thought of as having an actual, real, or "true" length; but that length can never be found completely, it can only be found *approximately*. How "exact" any particular measurement happens to be depends upon the nature of the instruments used, the skill of the operator, and the conditions under which it is made. The difference between the true length and the measured length is technically known as the *error*. An error is not a mistake. The careless use, or the misuse, of a measuring instrument leads to mistakes. The proper use of a measuring instrument always involves errors. The errors may be large or small; they can never be completely eliminated. The extent of the approximation is known as the *degree of accuracy* of the measurement; a numerical measure of the extent

of the error is known as the *precision* of the measurement. What particular degree of accuracy is sought depends chiefly upon the purpose for which the measurement is made, or the use to which the object is to be put. It would be a waste of time, for example, to measure the length of a fence post with the same care, i.e., with the same degree of accuracy, as used in measuring the length of a piston rod or a valve stem. Similarly, the accuracy used in measuring the length of a piece of molding in cabinet work would not suffice when measuring the diameter of needle valve. We shall learn more, however, about degrees of accuracy in Section 4 of this chapter.

Measurements are based upon certain *basic units* which are simply agreed upon by all concerned. In other words, the length of our present *inch* is what it is because it has been fixed by custom, or common consent; it has been *standardized,* so that it is always the same. But it might just as easily have been fixed, or standardized, at half its present length, or twice that amount, etc. In fact, at various times in history, and in different parts of the world, the inch has actually varied considerably from its present length. The same can be said of almost all the other units of measure that man has ever established or used: their values are *arbitrarily* fixed in accordance with conventional standards, and all who use these measures agree to abide by those standards.

**Kinds of Measures.** The fundamental measures are *length, mass,* and *time.* All other measures are derived from (or related to) one or another of these fundamental quantities. Thus *surfaces* (areas) and *volumes* (capacity) are measured in units derived from linear units; e.g., square inch and cubic inch. *Mass,* which is related to weight, is measured in the same units as weight, e.g., grams, ounces, pounds, etc. *Time* is measured in seconds, minutes, hours, etc. Other units of measure, such as those used for temperature, the electric current, or heat energy, are frequently more complicated. But whatever the unit used, it is always an arbitrary standard involving length, mass or time, or some combination of these. In this chapter we shall deal only with measures of length, area, capacity and weight.

**Standards of Measure.** Today all basic units of measure used in this country, such as length, weight, area, capacity, temperature, etc., are determined by the United States Bureau of Standards in Washington, D.C. In the main, they agree with similar units used throughout the world. Two major systems are in use,—the *English* system and the *Metric* system, although the latter is little used in industry and trade, being the tool primarily of the scientist.

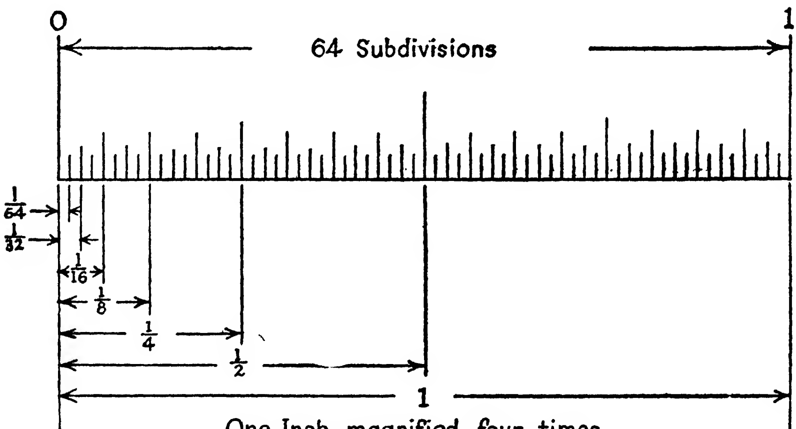
The best interests of all are served when the measuring instruments and devices used by scientists, engineers and industrial workers are sent to the Bureau of Standards at regular intervals to be checked for accuracy

against universally accepted standards. Furthermore, most trades and industries have set up the standards which are used in their own respective fields. These standards concern the choice of measuring units, various specifications, and working conditions; they do not, of course, conflict with the Federal Government's basic standards. But cooperation among various industries, and in various fields of technology, long ago became so important, that the formation of the American Standards Association was inevitable. This organization passes on the standards set in various parts of the country and works with similar organizations in other countries to establish and maintain desirable standards of measures, specifications, and trade practices.

**Linear Measure.** In the English system, the commonly used linear measures are as follows:

| | | | |
|---|---|---|---|
| | unit | = | 1 inch |
| 12 | inches | = | 1 foot |
| 3 | feet | = | 1 yard |
| 5½ | yards | = | 1 rod |
| 320 | rods | = | 1 mile |
| 1760 | yards | = | 1 mile |
| 5280 | feet | = | 1 mile |

The inch is sometimes subdivided by *halves,* sometimes by *tenths.* Each of these methods is here illustrated; both are commonly used by mechanics and industrial workers.




One Inch, magnified four times

When sub-divided by the decimal system, the subdivisions run as follows:

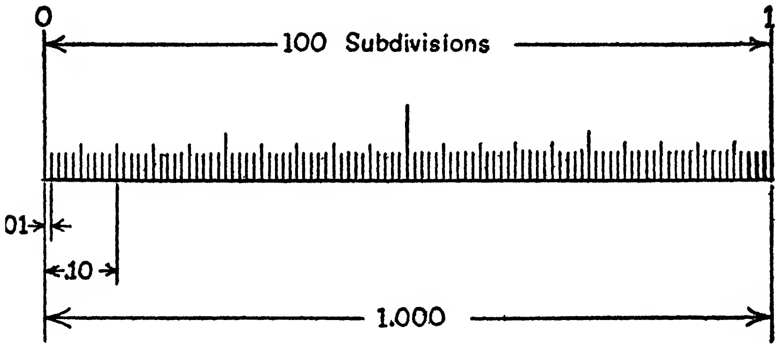
$1 = 1.000$ (one)

$\frac{1}{10} = .10$ (one-tenth)

$$\frac{1}{100} = .01 \quad \text{(one-hundredth)}$$

$$\frac{1}{1000} = .001 \quad \text{(one-thousandth)}$$

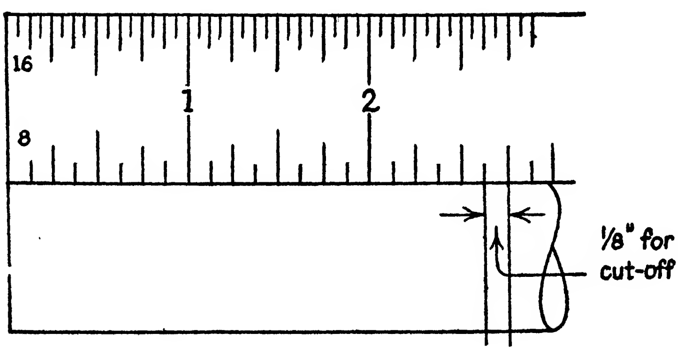
$$\frac{1}{10{,}000} = .0001 \quad \text{(one ten-thousandth)}$$



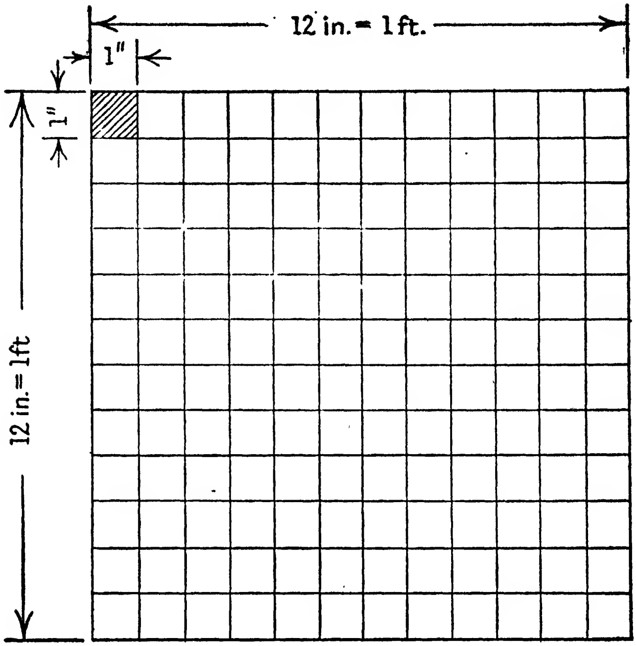

One Inch, magnified four times

**Surface Measure.** The commonly used measures of area in shop work and trade operations include:

| | |
|---|---|
| unit | =1 square inch |
| 144 square inches | =1 square foot |
| 9 square feet | =1 square yard |




Showing use of scale in measuring a 2½" piece to be cut from stock allowing 1/8" for cut-off.



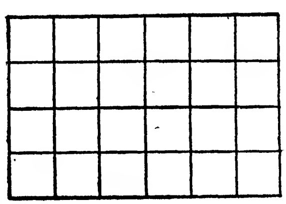

Pictorial representation, showing how
1 sq. ft. contains 144 sq. in.

A square inch is derived from the linear inch by taking as the unit of surface a square one inch on each side. The area of a surface is taken to be the number of square inches it contains. The area of a square is thus found by multiplying the length of one side by the other side, i.e.,

$$A = s \times s, \text{ or } A = s^2,$$

which is read *"area equals* s *squared."* Similarly, the area of a rectangle is found by multiplying its length by its width; i.e.,

$$A = l \times w, \text{ or } A = lw.$$

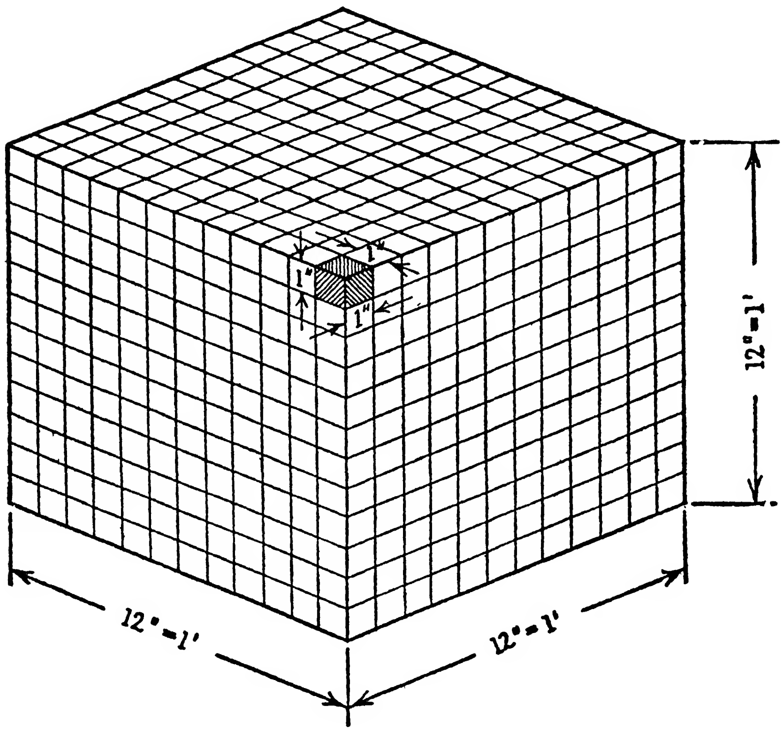


Area 4 x 6 = 24 Sq. in.

**Volume and Capacity.** The *volume* of a solid object such as a block of wood or a slab of metal refers to the amount of space it occupies. The *capacity* of a hollow vessel, such as a glass jar, a wooden box, or a metal bucket, refers to the quantity of material it can contain or hold; capacity is therefore expressed in terms of "inside" measurements, since the thickness of the walls of the container cannot be disregarded (unless they are

comparatively thin, or the measure of capacity is to be taken rather approximately). Volume and capacity are often measured in terms of the same units, viz.:

unit = 1 cubic inch
1728 cubic inches = 1 cubic foot
27 cubic feet = 1 cubic yard
1 cubic yard = 1 load




Pictorial representation, showing how 1 cu. ft. contains 1728 cu. in.

The unit of volume is taken as a cube each of whose edges is one inch in length. The volume of a cube is thus given by

$$V=e\times e\times e, \text{ or } V=e^3,$$

which is read *"volume equals e cubed."* Similarly, the volume of a rectangular solid of length $l$, width $w$ and thickness (or height) $h$, is given by

$$V=l\times w\times h, \text{ or } V=lwh.$$

For certain purposes, capacity is also measured in other units as well. For liquid measure, we have:

| | |
|---|---|
| 4 gills | =1 pint |
| 2 pints | =1 quart |
| 4 quarts | =1 gallon |

**Measures of Weight.** For measuring the weight of common objects, several systems of measures are in common use, viz., the *Avoirdupois, Troy,* and *Apothecaries'* systems. For all ordinary purposes, and unless otherwise specified, weights are in the Avoirdupois system; Troy weight is used only by jewelers, etc., when weighing gold, silver and gems, while the Apothecaries' system is used only by pharmacists and physicians.

AVOIRDUPOIS WEIGHT

| | |
|---|---|
| 16 ounces (oz.) | =1 pound (lb.) |
| 100 pounds | =1 hundredweight (cwt.) |
| 20 hundredweights | =1 ton (T.) |
| 2000 pounds | =1 ton |
| 2240 pounds | =1 long ton |
| 7000 grains (gr.) | =1 pound avoirdupois |

Certain common "equivalents" are well worth remembering. These are:

| | |
|---|---|
| 1 gallon | =231 cubic inches |
| 1 cubic foot | =7½ gallons |
| 1 barrel | =31½ gallons |
| 1 cubic foot of water | =62.4 pounds |

*Exercise 1.*

1. A roll of paper is 88″ wide; what is the width expressed in feet?
2. Find the length of the border of a rectangular rug measuring 9′3″×12′4″.
3. An airplane is traveling at a speed of 270 miles per hour; this is equivalent to how many feet per minute?
4. A piece of metal screening contains 20½ sq. yd.; how many square feet of surface will it cover?
5. A velocity of 88 ft. per second is equivalent to how many miles per hour?
6. A gallon of paint covers 150 sq. ft. with one coat. How many gallons will be required to cover a surface of 750 sq. ft. with two coats of this paint?
7. Linoleum costing 12¢ a sq. yd. was laid on a kitchen floor 12′×21′. What was the cost of the linoleum?
8. A point on the rim of a flywheel is moving at the rate of 1200 ft. per minute; what is the rate in feet per second?

9. A sheet of metal foil measures $2\frac{1}{2}'' \times 4''$. How many sq. ft. of foil are there in 1000 such sheets?
10. How many cubic feet are there to a gallon? to the barrel?
11. How much does a gallon of water weigh?
12. If a roof tank contains 1500 gallons of water, what is the weight of its contents?
13. A fuel tank for an oil burner has a capacity of 300 gallons; how many cubic feet is this?
14. A vat in a soap factory has a capacity of 1600 cu. ft. If it is filled to $\frac{3}{4}$ of its capacity with liquor weighing 70 lb. per cu. ft., what is the weight of its contents?
15. An air blower in a factory has a rectangular cross section, $12'' \times 20''$. Air is blown through the duct at the rate of 6 ft. per second. At this rate, how many cubic feet of air pass a given point every minute?

**Metric System.** This is the system of weights and measures used by scientists the world over. While it is legal and permissive in the United States, it has never been widely adopted in industrial and shop practice in this country.

The fundamental standard of length of the metric system is the meter, which is defined as the distance between two scratch-marks on a bar of platinum-iridium, carefully preserved at the International Bureau of Weights and Measures near Paris, when the temperature of the bar is that of melting ice (o° Centrigrade). All other units of length are multiples or submultiples of the meter, as shown below.

METRIC LINEAR MEASURE

| | |
|---|---|
| 10 millimeters (mm.) | =1 centimeter (cm.) |
| 10 centimeters | =1 decimeter (dm.) |
| 10 decimeters | =1 meter (m.) |
| 10 meters | =1 decameter (Dm.) |
| 10 decameters | =1 hectometer (Hm.) |
| 10 hectometers | =1 kilometer (Km.) |
| 10 kilometers | =1 myriameter (Mm.) |

It should be added that the most commonly used of these units are the kilometer, the meter, the centimeter and the millimeter. For purpose of converting lengths and distances from one system to the other, the following approximate equivalents are convenient:

APPROXIMATE EQUIVALENTS

| | |
|---|---|
| 1 centimeter | =about 0.4 inch |
| 1 meter | =about 1.1 yard |
| 1 kilometer | =about 0.6 mile |

| | |
|---|---|
| 1 inch | =about 2.5 centimeters |
| 1 yard | =about 0.9 meter |
| 1 mile | =about 1.6 kilometers |

**Metric Surface Measure.** The basic units of metric surface measure is the *square meter,* which is a square one meter long on a side. Another convenient and commonly used unit of area is the square centimeter, which is a square one centimeter on each side. The relations between the various units of area are given below.

METRIC SURFACE MEASURE

| | |
|---|---|
| 100 square millimeters (sq. mm.) | =1 square centimeter (sq. cm.) |
| 100 square centimeters | =1 square decimeter (sq. dm.) |
| 100 square decimeters | =1 square meter (sq. m.) |
| 100 square meters | =1 square decameter (sq. Dm.) |
| 100 square decameters | =1 square hectometer (sq. Hm.) |
| 100 square hectometers | =1 square kilometer (sq. Km.) |

For ordinary purposes and convenience in converting from English into metric units and vice versa, the following equivalents are given:

APPROXIMATE EQUIVALENTS

| | |
|---|---|
| 1 sq. inch | =about 6.5 sq. centimeters |
| 1 sq. foot | =about .09 sq. meter |
| 1 sq. yard | =about .84 sq. meter |
| 1 sq. mile | =about 2.6 sq. kilometers |
| 1 sq. centimeter | =about .15 sq. inch |
| 1 sq. kilometer | =about .39 sq. mi. |

**Metric Measures of Capacity.** In measuring cubical contents by the metric system the fundamental unit of volume is the *cubic meter.* Other units commonly employed are as follows:

METRIC UNITS OF VOLUME

| | |
|---|---|
| 1000 cubic millimeters (cu. mm.) | =1 cubic centimeter (cu. cm.) |
| 1000 cubic centimeters | =1 cubic decimeter (cu. dm.) |
| 1000 cubic decimeters | =1 cubic meter (cu. m.) |

For measuring both liquids and solids, the *liter* is commonly employed A liter is the same as a cubic centimeter, or 1000 cu. cm. (also abbreviated *cc.*).

APPROXIMATE EQUIVALENTS

| | |
|---|---|
| 1 cubic inch | =about 16 cubic centimeters |
| 1 cubic centimeter | =about .06 cubic inch |
| 1 quart (liquid) | =about .95 liter |
| 1 liter | =about 1.06 quart (liquid) |

**Metric Units of Weight.** Compared to the variety of English units of weight, the metric system is doubtless much simpler. Here the standard unit of weight is the *kilogram,* which is defined as the weight of a certain mass of platinum-iridium kept at the International Bureau of Weights and Measures near Paris, and known as the International Prototype Kilogram. The kilogram and the gram are the two most widely used units, except for very large weights.

METRIC MEASURES OF WEIGHT

| | |
|---|---|
| 10 milligrams (mg.) | =1 centigram (cg.) |
| 10 centigrams | =1 decigram (dg.) |
| 10 decigrams | =1 gram (g.) |
| 1000 grams | =1 kilogram (Kg.) |
| 1000 kilograms | =1 metric ton (T.) |

APPROXIMATE EQUIVALENTS

| | |
|---|---|
| 1 ounce=28.35 grams | 1 gram=.035 ounce |
| 1 pound=454 grams=.45 kilogram | 1 kilogram=2.2 pounds |

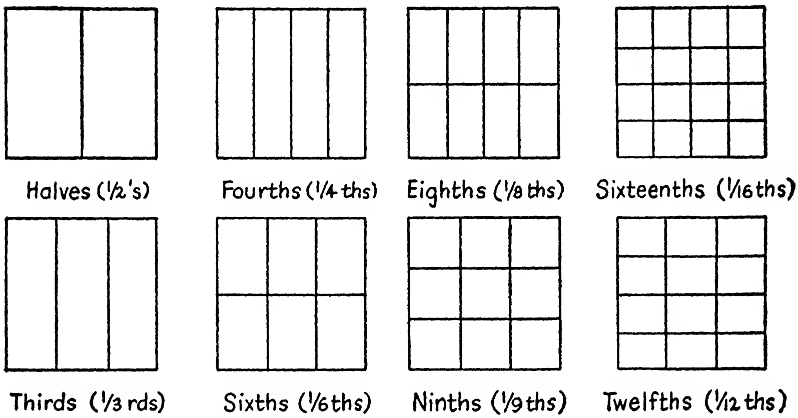
*Exercise 2.*

1. How many ounces are there in 250 grams?
2. How many inches are there in 45 millimeters?
3. How many centimeters are there in 15 inches?
4. How many kilograms are there in 12 pounds?
5. How many inches are there in 50 centimeters?
6. How many pounds are there in 75 kilograms?
7. How many millimeters are there in 6½ inches?
8. How many grams are there in 8 ounces?
9. What is the weight in kilograms of an athlete weighing 172 pounds?
10. What is the height in feet and inches of a man whose medical record states that he is 178 cm. tall?
11. How many cubic centimeters are there in a glass container capable of holding 2 quarts?
12. How many quarts can a 10-liter jug hold?
13. How many pints are there in a vessel whose capacity is 750 cc.?
14. How many liters are there in a 2-gallon chemical container?
15. How many sq. cm. are there in the surface of a metal plate having an area of 200 square inches?

## 2. COMMON FRACTIONS

**Whole Numbers and Fractions.** Numbers like 2, 5, 13, 40, etc., are known as whole numbers, or *integers.* When measurements are made, the magnitude is expressed, if possible, in some integral number of units as:

3 inches, 5 feet, 12 pounds, etc. But with many magnitudes this is impossible, and it becomes necessary to make use of a *part of a unit.* To do this, the entire unit is *thought of* as being divided into any convenient number of *equal parts;* one or more of such equal parts into which a whole unit has been divided is known as a *fraction.* Thus $\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{5}$ and $\frac{9}{10}$ are common fractions. The number below the fraction line, called the *denominator,* states into how many equal parts the whole unit has been divided; the number above the line, known as the *numerator,* tells how many of these equal parts are being considered, or measured, or used.




Fractions like $\frac{5}{8}$, $\frac{3}{16}$, $\frac{1}{4}$, and $\frac{15}{32}$, where the numerator is smaller than the denominator, are known as *proper fractions.* Proper fractions always indicate a quantity which is less than 1. On the other hand, fractions like $\frac{4}{3}$, $\frac{3}{2}$, $\frac{9}{8}$, $\frac{25}{12}$, where the numerator is greater than the denominator, are called *improper fractions;* such fractions always indicate an amount larger than 1, or one unit. Fractions like $\frac{2}{2}$, $\frac{3}{3}$, $\frac{8}{8}$, where the numerator equals the denominator, are also called improper fractions, although their value is exactly equal to 1 in each case. Strictly speaking, they do not represent a fractional part of a unit at all; they only have the form of a fraction, and actually represent the *whole unit,* since they say, in effect, "divide the whole unit into a certain number of equal parts, and then consider *all* of those parts."

An improper fraction of the first kind mentioned, e.g., $\frac{5}{3}$, can also be expressed as the sum of a whole number and a proper fraction; thus $\frac{5}{3}=\frac{3}{3}+\frac{2}{3}=1+\frac{2}{3}=1\frac{2}{3}$. When written in this final form, as an integer plus a proper fraction, but with the $+$ sign omitted, it is usually called a *mixed number.*

EXAMPLE 1: Express $^{25}/_{8}$ as a mixed number.

SOLUTION: $^{25}/_{8}=\dfrac{24+1}{8}={}^{24}/_{8}+{}^{1}/_{8}=3+{}^{1}/_{8}=3{}^{1}/_{8}$, *Ans.*

EXAMPLE 2: Change $4{}^{7}/_{16}$ to an improper fraction.

SOLUTION: $4{}^{7}/_{16}=4+{}^{7}/_{16}=\dfrac{4\times16}{16}+\dfrac{7}{16}={}^{64}/_{16}+{}^{7}/_{16}={}^{71}/_{16}$, *Ans.*

**Reduction of Fractions.** Proper fractions like $^{5}/_{8}$, $^{9}/_{16}$, $^{3}/_{4}$, and $^{2}/_{3}$ are said to be in their *lowest terms.* This means that both the numerator and the denominator cannot be further reduced by dividing each of them by the same number. On the other hand, fractions like $^{6}/_{8}$, $^{10}/_{16}$, $^{8}/_{32}$, and $^{4}/_{64}$ are *not* in their lowest terms; they can be reduced further by dividing *both* numerator and denominator by some number which is an exact divisor of each of them; dividing both the numerator and denominator of a fraction by a common divisor is always permissible, since it does not alter the value of the original fraction. Thus:

$$^{6}/_{8}={}^{3}/_{4};\quad {}^{10}/_{16}={}^{5}/_{8};\quad {}^{8}/_{32}={}^{1}/_{4};\quad {}^{4}/_{64}={}^{1}/_{16}.$$

Fractions can also be transformed by *multiplying* both numerator and denominator by any desired number, provided the *same* number is applied to both parts of the fraction; this does not change the value of the fraction either. It enables us, however, to express a fraction in any other denomination. Thus we have the following:

EXAMPLE 1: Change the fraction $^{7}/_{8}$ to an equivalent fraction having the denominator 32.

SOLUTION: Dividing 32, the new denominator, by 8, the old denominator, gives 4; this is the number by which we must multiply each part of the fraction.

$$^{7}/_{8}=\frac{7\times4}{8\times4}={}^{28}/_{32},\ \textit{Ans.}$$

EXAMPLE 2: Express $^{3}/_{4}$ as 64ths.

SOLUTION: $64\div4=16$

$$^{3}/_{4}=\frac{3\times16}{4\times16}={}^{48}/_{64},\ \textit{Ans.}$$

*Exercise 3.*

1. Change $^{3}/_{8}$ in. to 64ths; $2{}^{1}/_{4}$ in. to 16ths; $1{}^{5}/_{8}$ in. to 32nds; $10{}^{3}/_{16}$ in. to 64ths.

2. The thickness of a brass plate is $^{19}/_{32}''$; express this in eighths of an inch. Is its thickness more than $1/2''$? If so, how much more? Is it less than $3/4''$? If so, how much less?
3. Find the equivalent number of 16ths of an inch in $2\frac{1}{2}$ in.; in $1\frac{5}{8}$ in.; in $3\frac{3}{16}$ in.; in $3/4$ in.
4. A piece of cardboard is $5\frac{3}{4}$ in. wide. If it is to be cut into strips each $1/4$ in. in width, how many such strips will there be?
5. The face of a metal block is $3\frac{5}{8}$ in. wide; this is equivalent to how many 16ths? how many 32nds?
6. Is $^{19}/_{32}$ larger than $^{10}/_{12}$? Is $2\frac{1}{4}$ larger than $^{29}/_{16}$? Arrange the following dimensions in order of magnitude, beginning with the smallest: $7/8''$, $2\frac{1}{16}''$, $2\frac{3}{64}''$, $3/4''$.

**Addition and Subtraction of Fractions.** When adding two or more fractions having the same denominator, we simply add their numerators and place the result above their *common* denominator. Thus:

$$3/8+7/8+5/8=\frac{3+7+5}{8}={}^{15}/_{8}=1\tfrac{7}{8}.$$

If the denominators of the fractions to be added are not all alike, the fractions must first be changed to equivalent fractions which do have the same denominator. The *least common denominator* (L.C.D.) is the *smallest denominator* that is exactly divisible by *each* of the denominators in question. For example:

(1) to add $3/8$, $3/4$ and $1/2$, the L.C.D.$=8$

(2) to add $2/5$, $3/10$ and $1/4$, the L.C.D.$=20$.

In practical problems, the L.C.D. is readily found by inspection.

EXAMPLE 1: Add, $2/3+1/2+3/4$.

SOLUTION: L.C.D.$=12$.

$$2/3={}^{8}/_{12};\ 1/2={}^{6}/_{12};\ 3/4={}^{9}/_{12}$$

$$^{8}/_{12}+{}^{6}/_{12}+{}^{9}/_{12}=\frac{8+6+9}{12}={}^{23}/_{12}=1\tfrac{11}{12},\ \textit{Ans.}$$

EXAMPLE 2: Add, $2\frac{1}{4}+\frac{3}{16}+4\frac{7}{8}+\frac{5}{32}$

SOLUTION: L.C.D.$=32$.

$$^{9}/_{4}+{}^{3}/_{16}+{}^{39}/_{8}+{}^{5}/_{32}$$

$$={}^{72}/_{32}+{}^{6}/_{32}+{}^{156}/_{32}+{}^{5}/_{32}$$

$$=\frac{72+6+156+5}{32}={}^{239}/_{32}=7\tfrac{15}{32},\ \textit{Ans.}$$

**Example 3:** Subtract $4\frac{3}{8}$ from $6\frac{5}{16}$.

**Solution:** L.C.D.=16.

$$6\tfrac{5}{16}=6\tfrac{5}{16} \qquad 5\tfrac{21}{16}$$

$$4\tfrac{3}{8}=4\tfrac{6}{16} \qquad \underline{4\tfrac{6}{16}}$$

$$1\tfrac{15}{16}, \textit{ Ans.}$$

*Exercise 4.*

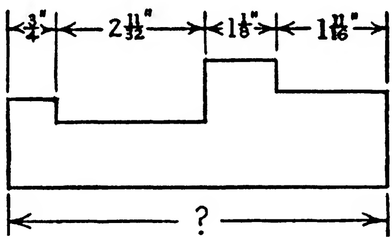
Add the following:

1. $\frac{2}{3}+\frac{3}{4}$
2. $\frac{1}{2}+\frac{5}{8}+\frac{1}{4}$
3. $\frac{1}{8}+\frac{7}{8}+\frac{3}{4}$
4. $\frac{3}{8}+\frac{1}{4}+\frac{5}{16}$
5. $2\frac{1}{2}+5\frac{7}{8}$
6. $3\frac{1}{4}+2\frac{1}{16}+1\frac{5}{8}$
7. $\frac{7}{8}+5\frac{1}{4}+2\frac{3}{16}$
8. $4\frac{1}{2}+\frac{5}{32}+2\frac{1}{8}+1\frac{3}{4}$

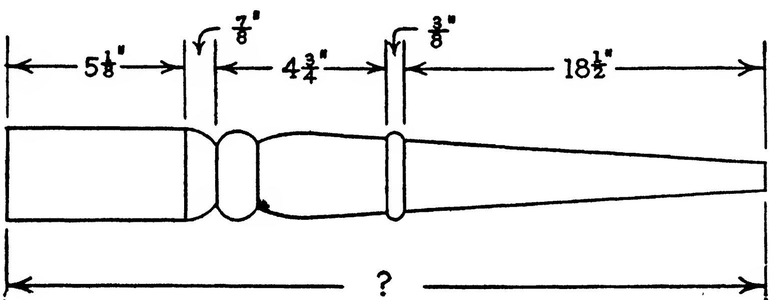
Subtract:

9. $3\frac{1}{2}$ from $5\frac{7}{8}$
10. $\frac{9}{16}$ from $2\frac{3}{4}$
11. $\frac{11}{64}$ from $\frac{9}{32}$
12. $3\frac{5}{8}$ from $5\frac{3}{32}$

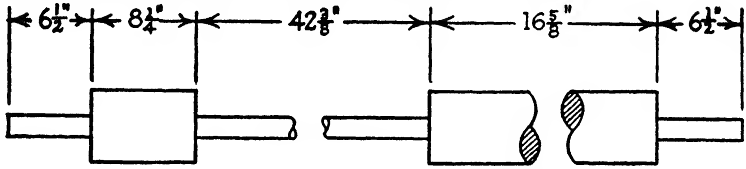
13. Find the overall length of the metal plate shown below:



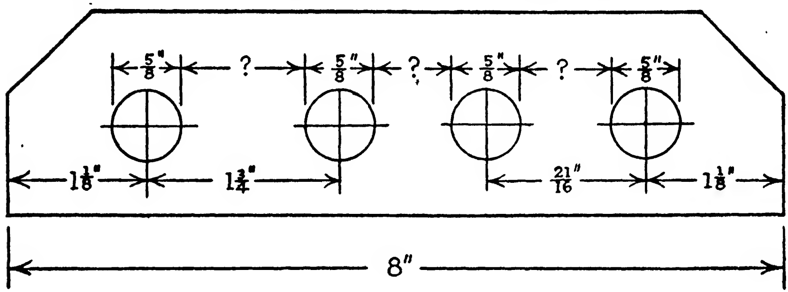

14. Find the total length of the wooden piece to be turned as follows:



15. What is the entire length of the shaft here shown?



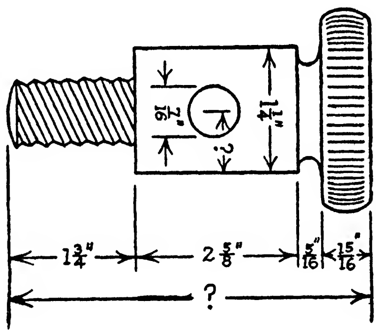

16. A metal plate is $1\frac{3}{16}''$ thick. If $\frac{1}{64}''$ is removed from the top surface and $\frac{3}{32}''$ from the bottom surface, what is its final thickness?

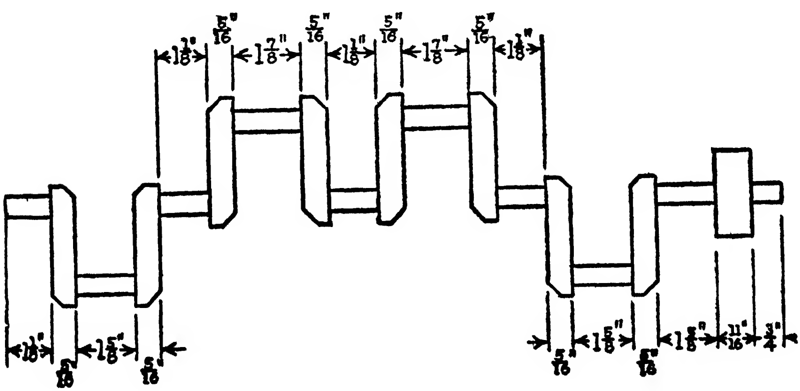



17. Find the missing dimensions in the wooden fixture with four holes drilled as per specification.

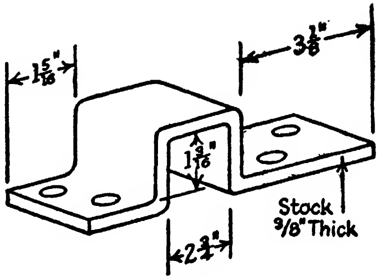
18. Find the overall length of the metal pin here shown; also, the distance from the center of the hole to the edge.

19. Find the overall length of the engine crank shaft here shown.



20. What length of stock is required to make the bent metal fixture here shown? (To allow for the bends, add ½ of the thickness of the stock for each right angle bend to the total of the inside measurements.)




**Multiplication and Division of Fractions.** In order to multiply two or more fractions together, the respective numerators are multiplied to find the numerator of the product, and their respective denominators are multiplied to find the denominator of the product; the resulting fraction is then reduced to lowest terms, if possible. Should some of the numerators have common factors, these should be divided out first (commonly called "cancellation";) the remaining factors are then multiplied together as before. If mixed numbers are to be multiplied, they are first changed to improper fractions.

Example 1: Multiply $\frac{3}{4}\times\frac{5}{8}\times\frac{1}{2}$.

Solution: $\frac{3}{4}\times\frac{5}{8}\times\frac{1}{2}=\frac{3\times5\times1}{4\times8\times2}=\frac{15}{64}$, *Ans.*

Example 2: Multiply $\frac{9}{16}\times\frac{2}{3}\times\frac{4}{5}$.

Solution: $\frac{\overset{3}{\cancel{9}}}{\underset{\underset{2}{\cancel{4}}}{\cancel{16}}}\times\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}}\times\frac{\overset{1}{\cancel{4}}}{5}=\frac{3\times1\times1}{2\times1\times5}=\frac{3}{10}$, *Ans.*

Example 3: Multiply $3\frac{1}{7}\times10\frac{1}{2}$.

Solution: $\frac{\overset{11}{\cancel{22}}}{\cancel{7}}\times\frac{\overset{3}{\cancel{21}}}{\cancel{2}}=33$, *Ans.*

To divide any quantity (a whole number, a fraction, or a mixed number) by a fraction, the divisor is inverted and then multiplied by the dividend; the same holds true even if the divisor is a whole number of an improper fraction. To divide by a mixed number, first change it to an improper fraction, and then proceed as before.

Example 1: Divide $\frac{4}{9}$ by $\frac{5}{6}$.

Solution: $\frac{4}{9}\div\frac{5}{6}=\frac{4}{\underset{3}{\cancel{9}}}\times\frac{\overset{2}{\cancel{6}}}{5}=\frac{8}{15}$, *Ans.*

EXAMPLE 2: Divide $\frac{9}{16}$ by 3.

SOLUTION: $\frac{9}{16} \div 3 = \frac{9}{16} \div \frac{3}{1} = \frac{\overset{3}{\cancel{9}}}{16} \times \frac{1}{\cancel{3}} = \frac{3}{16}$, *Ans.*

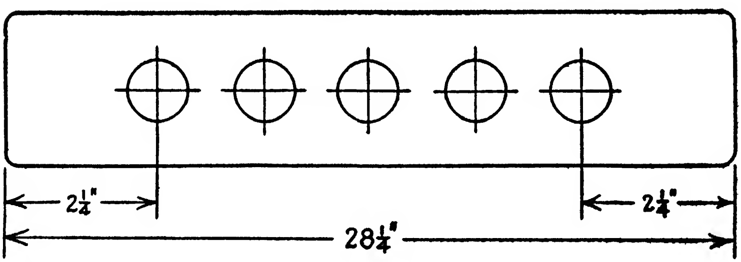
EXAMPLE 3. Divide $4\frac{2}{3}$ by $2\frac{1}{2}$.

SOLUTION: $4\frac{2}{3} \div 2\frac{1}{2} = \frac{14}{3} \div \frac{5}{2} = \frac{14}{3} \times \frac{2}{5} = \frac{28}{15} = 1\frac{13}{15}$, *Ans.*

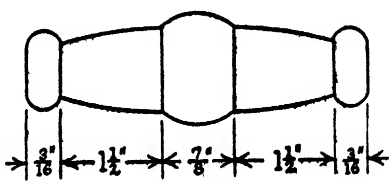
### *Exercise 5.*

Perform the following indicated multiplications or divisions:

1. $1\frac{3}{8} \times 2\frac{1}{2}$
2. $24 \times 3\frac{5}{8}$
3. $3\frac{1}{4} \times 1\frac{1}{2} \times 16$
4. $12\frac{3}{8} \div 2\frac{1}{4}$
5. $14 \div 1\frac{3}{4}$
6. $3\frac{3}{4} \div 6$
7. If there are $7\frac{1}{2}$ gal. in a cubic foot, how many gallons are there in a tank whose capacity is $28\frac{1}{2}$ cu. ft. when it is $\frac{3}{4}$ filled?
8. If the five holes bored in the wooden strip here shown are to have their centers equally spaced, find the center to center distance between holes.




9. Eight pins each $4\frac{3}{8}''$ in length are cut from a piece of stock $38\frac{1}{4}''$ long. Allowing $\frac{1}{16}''$ waste for each cut, what is the length of the piece left?
10. In turning wooden handles like this one shown, $\frac{3}{4}''$ is allowed for waste on each handle. How many such handles may be made from a piece of stock, if the stock comes in 8 ft. lengths?




11. What is the weight of six $14\frac{1}{2}$-ft. lengths of pipe, if this particular size pipe weighs $3\frac{1}{4}$ lb. per running foot?
12. How many circular discs each $\frac{5}{16}''$ thick can be cut from a metal rod $36\frac{1}{4}''$ long, if $\frac{3}{32}''$ waste is allowed for each cut?

## 3. DECIMAL FRACTIONS

**Meaning and Use of Decimal Fractions.** Fractions with a denominator of 10, 100, 1000, etc., may be written as *decimals,* i.e., without a fraction line and without expressing the denominator in numbers; thus .3 is the same as $\frac{3}{10}$, 0.16 is the same as $\frac{16}{100}$, and .247 is the same as $\frac{247}{1000}$. In all decimal fractions the denominators are multiples of 10; they do not have to be written out, since the position of the decimal point takes the place of the denominator. Ordinarily, these decimals would be read as follows:

.3 as "three tenths."
.16 as "sixteen hundredths."
.247 as "two hundred forty-seven thousandths."
.3859 as "three thousand eight hundred fifty-nine ten-thousandths."

In many trades, measurements are sufficiently accurate when expressed in common fractions of an inch. However, not all fractional measurements in shop work are expressed as common fractions; as a matter of fact, in the machine shop, especially in precision work, the machinist uses decimal fractions of an inch more often than common fractions. The machinist, who uses precision instruments to attain the fine measurements called for, has developed a method of reading and designating decimals which differs somewhat from the expressions used by non-technical people. Studying the table given below will show you how to "talk decimals" in the machine-shop manner:

| *Decimal* | *Designation.* |
|---|---|
| 0.0001 | One-tenth thousandth. |
| 0.00025 | One-quarter thousandth. |
| 0.0005 | One-half thousandth. |
| 0.00075 | Three-quarter thousandth. |
| 0.001 | One thousandth. |
| 0.00125 | One and one-quarter thousandths. |
| 0.0015 | One and one-half thousandths. |
| 0.002 | Two thousandths. |
| 0.0025 | Two and one-half thousandths. |
| 0.003 | Three thousandths. |
| 0.0075 | Seven and one-half thousandths. |
| 0.010 | Ten thousandths. |
| 0.0125 | Twelve and one-half thousandths. |
| 0.015 | Fifteen thousandths. |
| 0.0156 | Fifteen and six-tenth thousandths. |
| 0.0312 | Thirty-one and two-tenth thousandths. |
| 0.1718 | One hundred seventy-one and eight-tenth thousandths. |

*Exercise 6.*

Write each of the following in figures:

1. Two hundred and seventy-six thousandths.
2. Fifteen and four-tenth thousandths.
3. Seven-tenth thousandths.
4. Four and one-quarter thousandths.
5. One hundred thousandths.

Write each of the following in words:

| | | |
|---|---|---|
| 6. 0.3792 | 9. 0.0705 | 12. 0.4444 |
| 7. 0.0006 | 10. 0.2816 | 13. 0.0158 |
| 8. 0.2002 | 11. 0.0960 | 14. 0.00025 |

**Addition and Subtraction of Decimals.** These operations are carried out exactly as in the case of whole numbers, an important feature being the decimal points, which must be kept one under the other, including the decimal point in the sum or difference.

EXAMPLE 1: Add: 0.316+0.0592+1.8034+.26

SOLUTION:
$$\begin{array}{l} 0.316 \\ 0.0592 \\ 1.8034 \\ 0.26 \\ \hline 2.4386, \textit{ Ans.} \end{array}$$

EXAMPLE 2: From 124.307, subtract 88.092.

SOLUTION:
$$\begin{array}{r} 124.307 \\ 88.092 \\ \hline 36.215, \textit{ Ans.} \end{array}$$

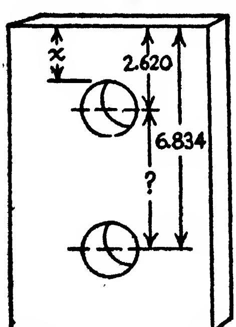
Another point that should be remembered with regard to adding or subtracting *measurements* involving decimals: never add or subtract measurements having different numbers of decimal places. Always "round off" the measurements, as required, to the same number of decimal places as appear in the measurement having the least number of decimal places, as shown below:

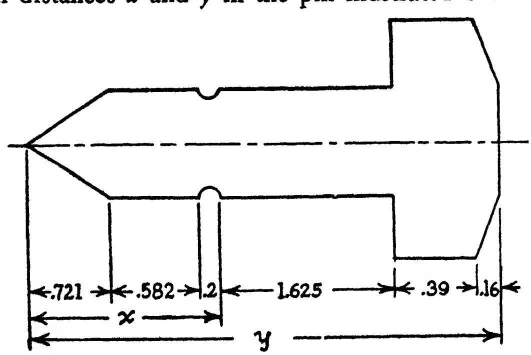
| *Incorrect* | | *Correct* | |
|---|---|---|---|
| 8.36 | inches | 8.4 | inches |
| 10.082 | " | 10.1 | " |
| 4.5928 | " | 4.6 | " |
| 7.8 | " | 7.8 | " |
| .749 | " | .7 | " |
| 31.5838 | inches | 31.6 | inches |

If, however, we wish to add a measurement of 14¾″ to another measurement of 23½″, it is correct to say 14.75″+23.5″=38.25″, *provided that the second measurement is also taken accurately to two decimal places,* i.e., correct to the nearest .01 inch, in which case it should have been more properly expressed as 23.50″, showing that both the tenth's and hundredth's place had *actually been measured.* An exception to this statement may be made wherever dimensions on a drawing, or actual measurements on a piece of work, are understood to be taken to the same degree of accuracy; also when using precision gage blocks, as explained later in Section 4 of this chapter.

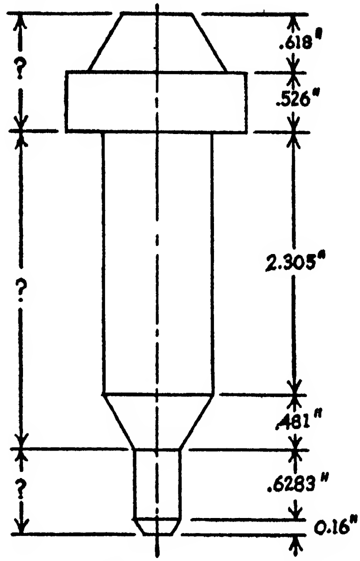
*Exercise 7.*

1. A hollow metal cylinder has a measured inside diameter of 2.0275″. If the wall of the cylinder is .245″ thick, what is the outside diameter?
2. A round piece of work is supposed to have a diameter of 2.375″. If it was turned .0008″ too large, what was the diameter?
3. Because of a mistake in dimensions on a blueprint a piece of work measuring 1.428 in. in thickness must be reduced 0.236 in. What will the thickness be after the reduction has been made?
4. It is desired to grind down a flat steel plate measuring 0.625″ thick by taking off .0075″. What will it measure after it has been ground?
5. A machinist milled off 0.184″ from each face of a circular brass plate. Before he had done this the plate measured 1.062″ in thickness. What did it measure afterward?
6. A toolmaker in checking a precision measurement uses four gage blocks measuring, respectively, as follows: .141″, .250″, 1.0007″ and 3.000″. Find the combined thickness of the four blocks.
7. Find the inside diameter of a circular tube whose outside diameter is 1.804″ and whose thickness is .216″.
8. Find the center distance between the two holes in the plate shown below. If the diameter of each hole is 1.62″, find the distance *x*.
9. Find the two overall distances *x* and *y* in the pin illustrated below.



10. The actual diameter of a crank shaft is 4.2836″. The inside diameter of the bearing into which this crank shaft fits is found to be 4.285″. What clearance does the shaft have?
11. The inside diameter of a hollow shaft measures 3.026″, and the outside diameter, 3.504″. Find the thickness of the shaft.
12. Find the indicated missing dimensions in the piece shown at the right.




**Multiplication of Decimals.** When a whole number is to be multiplied by a decimal, or when two decimals are to be multiplied together, the multiplication is first carried out exactly as with whole numbers; then, beginning at the right of the product, point off as many decimal places as there are in both factors together, prefixing ciphers if necessary.

EXAMPLE 1: Multiply 3.1416 by 32.

SOLUTION:
```
  3.1416
      32
  ------
   62832
  94248
  --------
100.5312, Ans.
```

EXAMPLE: 2: Multiply .592 by .013.

SOLUTION:
```
  .592
  .013
  ----
  1776
  592
  -------
.007696, Ans.
```

*Exercise 8.*

1. The length of the side of any square is always equal to the length of its side multiplied by 1.414. How long is the diagonal of a square 4.21 inches on a side?

2. The specific gravity of cast iron is 7.13, which means that cast iron is 7.13 times as heavy as an equivalent volume of water. If water weighs 62.425 lb. per cu. ft., what is the weight of 1 cu. ft. of cast iron?
3. A sheet of newspaper is .0031″ in thickness. What is the approximate thickness of a Sunday newspaper consisting of 220 pages?
4. The horizontal component of a force acting at an angle of 30° to the horizontal is equal to the force multiplied by .866. If the force amounts to 250 lb., what is the amount of the horizontal component?
5. A laminated piece is built up of 45 pieces of metal, each piece having a thickness of .0024″. How thick is the entire piece?
6. Gasoline is .91 times as heavy as water. If a quart of water weighs 2.08 lb., what is the weight of a gallon of gasoline?
7. A certain broach has 36 teeth. If each tooth cuts .018″, how much material is removed by the entire broach?
8. The specific heat of aluminum equals 0.218; this represents the number of calories required to raise 1 gm. of aluminum 1 degree Centigrade. How many calories are required to heat an aluminum block weighing 8.04 gm. from 20.4° C to 100.1° C?
9. A certain size steel bar weighs 7.656 lb. per linear foot. Find the cost of 2000 ft. of this bar, if the price is $2.20 per 100 lb.
10. The coefficient of expansion of iron equals .00000672 per degree Fahrenheit, which means that for each degree rise in temperature it increases in length by that fractional part of its original length. By how much will the length of a 200 ft. iron cable increase if the temperature rises 80° F.?

**Division of Decimals.** In order to divide a number by a decimal, the decimal point in both the divisor and the dividend must be moved as many places to the right as there are decimal places in the divisor.

Example: Divide 92.862 by 2.91.

Solution:

```
       31.91
  291)9286.20          Ans., 31.91+
      873
      ---
       556
       291
       ----
       2652
       2619
       ----
         330
         291
         ---
```

*Exercise 9.*

1. If one cubic foot contains 7.48 gal., find, to the nearest tenth, the number of cubic feet occupied by 550 gallons.

2. A certain size metal rod weighs 2.84 lb. per linear foot. How many feet of these rods are there in a bundle of various lengths, if the entire bundle weighs 86.75 lb.?
3. There are .3937 inches to one centimeter. How many centimeters long is a wire measuring 6.54″? (Carry the result to two decimal places.)
4. How many metal discs 0.0625″ thick can be stacked to a height of 2¼″?
5. A steel rod 38.26″ long is to be cut into 7 equal parts. Allowing 0.032″ for the thickness of each cut, how long will each piece be? (Remember that only six cuts need be made.)
6. In order to find the number of screws in a box, a mechanic weighs the entire box full of screws and finds the weight to be 2¼ lb. He also finds that a dozen screws weigh .28 lb. Making no allowance for the weight of the box, how many screws does it contain?

**Changing Common Fractions to Decimals.** Any common fraction may readily be changed to an equivalent decimal fraction simply by annexing zeros to the numerator and dividing by the denominator, as shown below; if the division does not terminate, it may be carried to as many decimal places as desired.

Example 1: Change 7/23 to a decimal.

Solution:

```
       .30434
   23)7.00000          Ans., .30434+
      6 9
        100
         92
          80
          69
          110
           92
```

Example 2: Reduce 9/32 to an equivalent decimal.

Solution:

```
       .28125, Ans.
   32)9.0000
      6 4
      2 60
      2 56
         40
         32
          80
          64
          160
          160
```

**Changing Decimals to Common Fractions.** To change a given decimal to a common fraction it is merely necessary to rewrite it with the denominator expressed as 10, 100, 1000, 10,000, etc., and reduce it to lowest terms if possible.

EXAMPLE 1: Change .042 to a common fraction.

SOLUTION: $.042 = \frac{42}{1000} = \frac{21}{500}$, *Ans.*

EXAMPLE 2: Reduce .6784 to a common fraction.

SOLUTION: $.6784 = \frac{6784}{10,000} = \frac{1696}{2500} = \frac{424}{625}$, *Ans.*

*Exercise 10.*

Change each of the following to decimal fractions:

| | | | |
|---|---|---|---|
| 1. $\frac{1}{8}$ | 5. $\frac{9}{32}$ | 9. $3\frac{1}{7}$ | 13. $\frac{11}{64}$ |
| 2. $\frac{3}{32}$ | 6 $\frac{15}{16}$ | 10. $\frac{5}{9}$ | 14. $\frac{49}{64}$ |
| 3. $\frac{7}{16}$ | 7. $\frac{27}{32}$ | 11. $\frac{12}{13}$ | 15. $\frac{5}{24}$ |
| 4 $\frac{15}{64}$ | 8. $\frac{11}{16}$ | 12. $\frac{2}{11}$ | 16. $\frac{25}{128}$ |

Change each of the following to common fractions:

| | | |
|---|---|---|
| 17. .4375 | 20. .003125 | 23. .078125 |
| 18. .8125 | 21. .40625 | 24. .921875 |
| 19. .0625 | 22. .90625 | 25. .531250 |

**Table of Decimal Equivalents.** Since measurements on blueprints and in the shop are commonly expressed both as decimals and as ordinary fractions, it is important to be able to convert from one to the other quickly and easily. It is therefore highly desirable that some of the decimal equivalents should be memorized, especially the following:

| | |
|---|---|
| $\frac{1}{2} = 0.500$ | $\frac{1}{16} = 0.0625$ |
| $\frac{1}{4} = 0.250$ | $\frac{1}{32} = 0.03125$ |
| $\frac{1}{8} = 0.125$ | $\frac{1}{64} = 0.0156$ |

For further convenience the Table of Decimal Equivalents given herewith is constantly used by draftsmen and machinists.

**Tolerance.** Architects, carpenters and patternmakers generally use eighth's, sixteenth's, and thirty-second's instead of decimals. But they use them to different degrees of accuracy; thus the patternmaker rarely uses fractions of less than $\frac{1}{16}''$; the carpenter or cabinetmaker generally does not use fractions of less than $\frac{1}{8}''$; and so on. However, when it comes to making dies, tools and machine parts, great accuracy is required; for this purpose decimal fractions are more convenient and more useful. The term *tolerance* is used to indicate the limits within which a piece of work is acceptable when it deviates from the dimension indicated.

## TABLE OF DECIMAL EQUIVALENTS

| | | | |
|---|---|---|---|
| 1/64 | .015625 | 33/64 | .515625 |
| 1/32 | .03125 | 17/32 | .53125 |
| 3/64 | .046875 | 35/64 | .546875 |
| 1/16 | .0625 | 9/16 | .5625 |
| 5/64 | .078125 | 37/64 | .578125 |
| 3/32 | .09375 | 19/32 | .59375 |
| 7/64 | .109375 | 39/64 | .609375 |
| 1/8 | .1250 | 5/8 | .6250 |
| 9/64 | .140625 | 41/64 | .640625 |
| 5/32 | .15625 | 21/32 | .65625 |
| 11/64 | .171875 | 43/64 | .671875 |
| 3/16 | .1875 | 11/16 | .6875 |
| 13/64 | .203125 | 45/64 | .703125 |
| 7/32 | .21875 | 23/32 | .71875 |
| 15/64 | .234375 | 47/64 | .734375 |
| 1/4 | .2500 | 3/4 | .7500 |
| 17/64 | .265625 | 49/64 | .765625 |
| 9/32 | .28125 | 25/32 | .78125 |
| 19/64 | .296875 | 51/64 | .796875 |
| 5/16 | .3125 | 13/16 | .8125 |
| 21/64 | .328125 | 53/64 | .828125 |
| 11/32 | .34375 | 27/32 | .84375 |
| 23/64 | .359375 | 55/64 | .859375 |
| 3/8 | .3750 | 7/8 | .8750 |
| 25/64 | .390625 | 57/64 | .890625 |
| 13/32 | .40625 | 29/32 | .90625 |
| 27/64 | .421875 | 59/64 | .921875 |
| 7/16 | .4375 | 15/16 | .9375 |
| 29/64 | .453125 | 61/64 | .953125 |
| 15/32 | .46875 | 31/32 | .96875 |
| 31/64 | .484375 | 63/64 | .984375 |
| 1/2 | .5000 | 1. | 1.0000 |

**EXAMPLE 1:** If the diameter of a round piece as indicated is 2.138″, with a tolerance of ±.003″; what are the limits within which it is acceptable?

**SOLUTION:**

$$\left.\begin{array}{l}2.138+.003=2.141\\2.138-.003=2.135\end{array}\right\}\textit{Ans.}$$

Thus any dimension less than 2.141″ and more than 2.135″ is acceptable. Such a dimension might be written as 2.138±.003.

**Example 2:** What are the limits of measurement on a piece that calls for $3\frac{5}{16}''$, plus or minus $.010''$?

**Solution:** $3\frac{5}{16}=3.3125$

$$\left.\begin{array}{l}3.3125+.01=3.3135''\\3.3125-.01=3.3115''\end{array}\right\}\textit{Ans.}$$

**Example 3:** Using the table of decimal equivalents, change a fractional measurement of $.714''$ to the nearest 32nd of an inch.

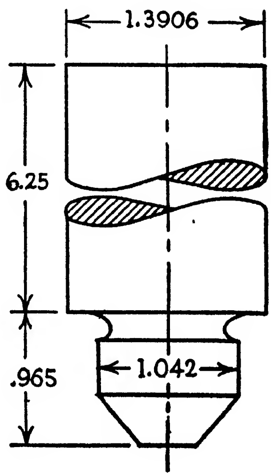
**Solution:** From the table,

$\frac{23}{32}=.71875$
$\frac{22}{32}=\frac{11}{16}=.6875$
$\frac{24}{32}=\frac{3}{4}=.75$

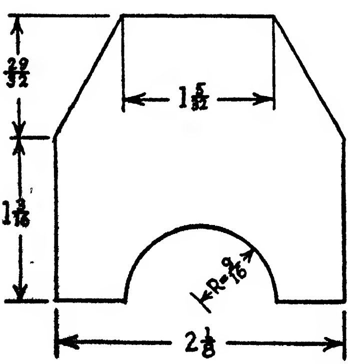
thus $.714=\frac{23}{32}$, approx., *Ans.*

*Exercise 11.*

1. Using the table, change the following decimal dimensions to the nearest 32nd of an inch:
   a) .216 b) .8391 c) .10123
2. Change the following to the nearest 64th of an inch:
   a) .4869 b) .17235 c) .89268
3. In the round piece shown, find the allowable limits for each of the given dimensions if the tolerance is $\pm.004$.
4. Find the limits for each dimension of the plate shown, if the tolerance allowed is $\pm.025$.



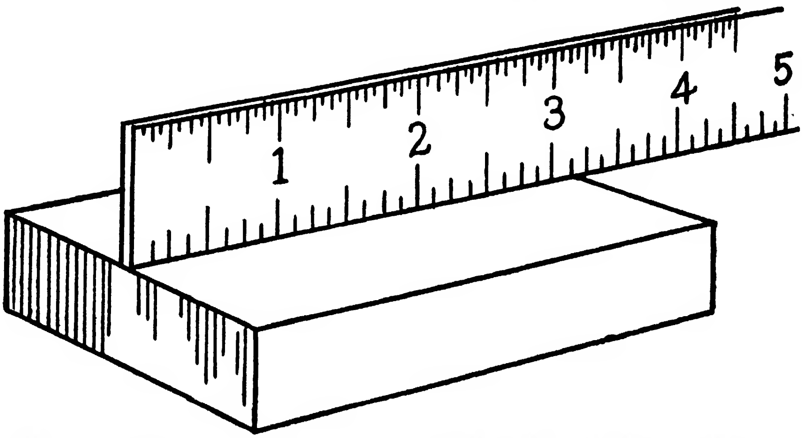
Ex. 3



Ex. 4

## 4. MEASURING INSTRUMENTS

**Degree of Accuracy.** As we saw in Section 1, all measurements are approximations, and the degree of accuracy of measurements may vary considerably. Extreme accuracy is not always required in shop work, even in machine shop operations. The greater the degree of accuracy achieved, the greater is the cost of the operation; it is uneconomical to secure greater accuracy than is actually needed. Where ultimate extreme accuracy is de-




sired, the degree of accuracy varies with the successive steps in the operation. Thus rough machining, finishing machining, grinding and lapping are often used in succession to achieve a final high degree of accuracy. For rough machining a steel scale would be used for making the measurement, which would be made to within $\pm 1/32''$. For the second, or finishing machining, a micrometer might be used, reading the measurement to $\pm 1/64''$. For the grinding operation a precision micrometer would be used, graduated in .0001″, and the measurement would be taken to within $\pm .0002''$. During the lapping operation and for the finished part, gage blocks and indicators would be employed.

**Limits of Accuracy.** Scientists often use extremely precise measurements; the wave length of sodium light, for example, is 0.00005893 cm., which represents accuracy to 8 decimal places. For most industrial and shop operations, however, the limit of accuracy required is usually to the fourth decimal place. As we have already seen, the limits of accuracy required are referred to as the *tolerances,* and are generally specified on the blueprint or in the specifications which accompany the designs; they are just as important as the dimensions or measurements themselves. The American Standards Association has standardized tolerances for various parts, such as screw threads, cylindrical fits, and surface finish.

The limits of accuracy obtainable in making a measurement depend upon the nature of the particular instrument used, the conditions under which the measurement is made, and the skill of the operator. So far as the possible limitations of the measuring instrument are concerned, the following should be noted. The common steel scale has a limit of 1/64″ or 1/100″. The micrometer caliper will yield an accuracy of .001″, and, with a vernier attachment, to .0001″. These figures represent graduations on the scales of the instruments. The use of a toolmaker's miscroscope, or magnifying glass, is sometimes required when measuring to a graduated line, since the width of the line itself is approximately .006″. Ordinary precision gage blocks measure to .000008 of an inch, and the finest grade blocks to .000002 of an inch. Such blocks have no graduations, but are fixed in measurement; they are used in combination as will be explained below.

**Conditions under Which Instruments Are Used.** The degree of accuracy obtainable depends also upon the quality of the instrument and the working conditions. A particular instrument, even when new, will vary in manufactured accuracy, depending upon the quality and cost of the tool; an inexpensive, poorly made scale or calipers is never as accurate as a high grade, carefully made instrument. Again, as a tool becomes worn with constant use, it loses in accuracy. The lines marking the graduations on the scale become obliterated, the edge becomes nicked, and the moving parts (if any) become loose, and bearings are thrown out of alignment by knocking; all of these conditions diminish the accuracy obtainable. Finally, when working to four-place accuracy, temperature variations affect the accuracy of a measurement, since with metals particularly, both the object and the instrument are subject to expansion and contraction with a rise or fall in temperature. Thus for very accurate work the temperature should be ordinary room temperature, i.e., from about 68° to 72° Fahrenheit.

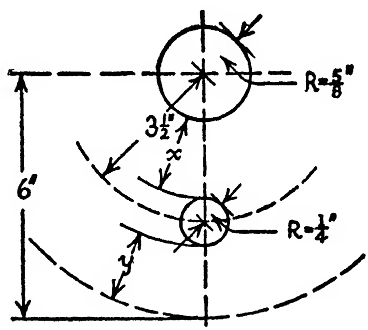
**The Human Equation.** Personal factors also influence accuracy. These include such considerations as normal eyesight and proper lighting; skill and care in estimating the smallest subdivision of a scale; correct habits, such as reading a meniscus properly, or avoiding parallax; and a delicate sense of touch, the ability to "feel" measure on a measuring instrument. Meticulous care in handling tools, their skilful use, and dependable judgment that comes only with experience—all these are required in making careful measurements. Lathes, shapers, milling machines, drills, taps, etc., are so designed and constructed that tolerances of ±.001″ or better can be achieved if skill and care are used. When extreme accuracy is required, unusual care must be exercised. Thus if a micrometer jaw is set too tightly, as much as .0005″ can be "forced"; or again, when reading a vernier scale, undue pressure against the sliding jaw may cause deflec-

tion of the object being measured, or slight clearances in the bearing surfaces of the sliding jaw, either of which will affect the reading of the instrument adversely.

**Mechanical Duplication.** The possibility of securing mechanical duplication depends not only upon the measuring instruments and the skill of the operator, but also upon the materials used and the machines and tools involved. Stock varies in quality and composition, in texture and finish, according to the particular shipment; such variations increase the difficulties in producing parts exactly alike. Similarly with equipment: the rigidity of a machine, the tightness of the bearings, the solidity of the machine bed, the fastening of fixtures, the wear and support of tools —all these are also involved in mechanical duplication.

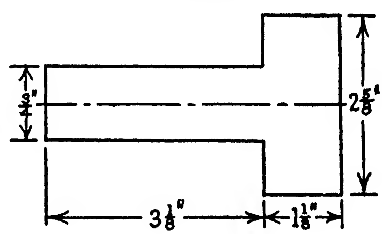
*Exercise 12.*

1. How much must be removed from a metal piece to be turned, if the original diameter is 3.764″ and the finished diameter required is 2.856″? if the original diameter is 1.827″ and the finished diameter is 1.792″?
2. Find the distances $x$ and $y$, respectively, *before* the holes are reamed. If 1⁄64″ is allowed for reaming, what are these distances *after* reaming?




Ex. 2

3. The piece shown is to be rough turned, allowing 1⁄16″ for finish machining on all diameters and 1⁄32″ on all faces; find the corresponding dimensions for rough turning.



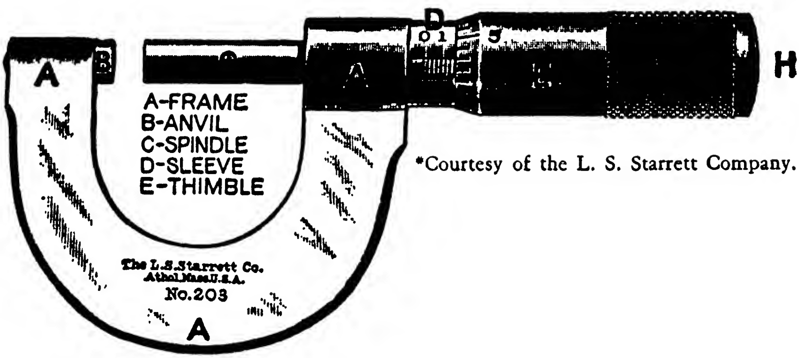

Ex. 3

4. The same piece (Ex. 3) is to be finish turned to allow .012″ for grinding on all diameters and .008″ on all faces; find the corresponding dimensions for finish machining.
5. If the required dimensions of a finished piece of work must be .462±.002, and the piece now measures .467, how much must still be removed?

**The Micrometer.** The *micrometer caliper,* or "mike," is the most common "precision instrument" used in the machine shop. Every machinist and toolmaker carries a micrometer. This instrument has many advantages:

1. It is small, and easily carried in the pocket.
2. It is convenient to handle and easy to read.
3. It is rugged enough to stand considerable handling.
4. It retains its accuracy, and has adjustments to compensate for wear.
5. It has a practical range of measurement, generally up to one inch.
6. It is comparatively inexpensive.

**Construction and Use of the Micrometer.*** The spindle C is attached to the thimble E, on the inside, at the point H. The part of the spindle which is concealed within the sleeve and thimble is threaded to fit a nut in the frame A. The frame being held stationary, the thimble E is revolved by the thumb and finger, and the spindle C, being attached to the thimble, revolves with it, and moves through the nut in the frame, approaching or receding from the anvil B. The article to be measured is placed between the anvil B and the spindle C. The measurement of the opening between the anvil and the spindle is shown by the lines and figures on the sleeve D and the thimble E.




*Courtesy of the L. S. Starrett Company.

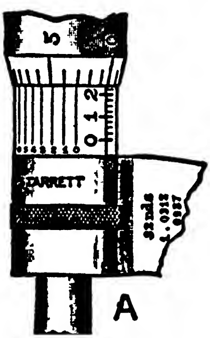
The pitch of the screw threads on the concealed part of the spindle is 40 to an inch. One complete revolution of the spindle therefore moves it longitudinally one-fortieth (or twenty-five thousandths) of an inch. The sleeve D is marked with 40 lines to the inch, corresponding to the number of threads on the spindle. When the caliper is closed, the beveled edge of the thimble coincides with the line marked 0 on the sleeve, and the 0 line on the thimble agrees with the horizontal line on the sleeve. Open the caliper by revolving the thimble one full revolution, or until the 0 line on the thimble again coincides with the horizonal line on the sleeve; the distance between the anvil B and the spindle C is then $1/40$ (or .025) of an inch, and the beveled edge of the thimble will coincide with the second

vertical line on the sleeve. Each vertical line on the sleeve indicates a distance of 1/40 of an inch. Every fourth line is made longer than the others, and is numbered 0, 1, 2, 3, etc. Each numbered line indicates a distance of four times 1/40 of an inch, or one-tenth.

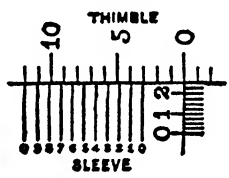
The beveled edge of the thimble is marked in twenty-five divisions, and every fifth line is numbered from 0 to 25. Rotating the thimble from one of these marks to the next moves the spindle longitudinally 1/25 of twenty-five thousandths or one-thousandth of an inch. Rotating it two divisions indicates two thousandths, etc. Twenty-five divisions will indicate a complete revolution, .025 or 1/40 of an inch.

To read the caliper, therefore, multiply the number of vertical divisions visible on the sleeve by 25, and add the number of divisions on the bevel of the thimble, from 0 to the line which coincides with the horizontal line on the sleeve. For example, as the tool is represented in the engraving, there are seven divisions visible on the sleeve. Multiply this number by 25, and add the number of divisions shown on the bevel of the thimble, 3. The micrometer is open one hundred and seventy-eight thousandths. ($7 \times 25 = 175 + 3 = 178$.)

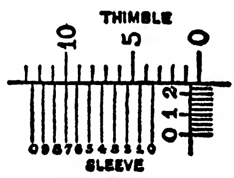
**Using a Micrometer Graduated in Ten-Thousandths of an Inch.** Readings in ten-thousandths of an inch are obtained by the use of a vernier, so named from Pierre Vernier, who invented the device in 1631. As applied to a caliper this consists of ten divisions on the adjustable sleeve, which occupy the same space as nine divisions on the thimble. The difference between the width of one of the ten spaces on the sleeve and one of the nine spaces on the thimble is therefore one-tenth of a space on the thimble. In engraving B the third line from 0 on thimble coincides with the first line on the sleeve. The next two lines on thimble and sleeve do not coincide by one-tenth of a space on thimble; the next two, marked 5 and 2, are two-tenths apart, and so on. In opening the tool, by turning the thimble to the left, each space on the thimble represents an opening of one-thousandth of an inch. If, therefore, the thimble be turned so that the lines marked 5 and 2 coincide, the caliper will be opened



A




B

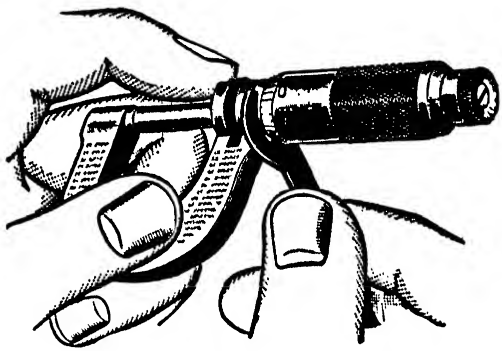



C

two-tenths of one-thousandth or two ten-thousandths. Turning the thimble further, until the line 10 coincides with the line 7 on the sleeve, as in engraving C, the caliper has been opened seven ten-thousandths, and the reading of the tool is .2507.

To read a ten-thousandths caliper, first note the thousandths as in the ordinary caliper, then observe the line on the sleeve which coincides with a line on the thimble. If it is the second line, marked 1, add one ten-thousandth; if the third, marked 2, add two ten-thousandths, etc.

**Adjusting the Micrometer.** These calipers will read correctly if there is no dirt between the anvil and spindle. When it becomes necessary to readjust the tool to compensate for the wear of screw and nut, this is done, not by the anvil, but by means of our friction sleeve, as follows: Take up the wear of screw and nut, then remove all dirt from face of the anvil and spindle and bring them together carefully. Insert the small spanner wrench in the small hole and turn until the line on the sleeve coincides with the zero line on the thimble.



*Exercise 13.*

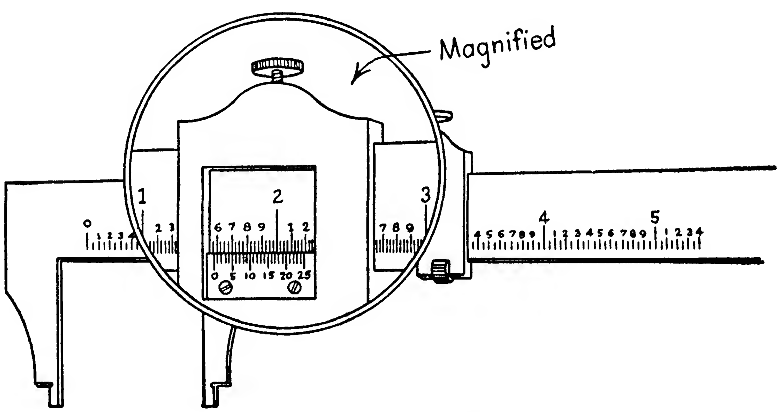
1. The following readings are taken on an ordinary micrometer; complete the table:

| | Reading on sleeve is between | Nearest line on thimble | Complete Reading |
|---|---|---|---|
| (a) | .450 and .475 | 15 | |
| (b) | .900 and .925 | 4 | |
| (c) | .575 and .600 | 20 | |
| (d) | .125 and .150 | 7 | |
| (e) | .000 and .025 | 22 | |
| (f) | .975 and 1.000 | 16 | |

2. The following readings are taken on a micrometer graduated in ten-thousandths of an inch; complete the table:

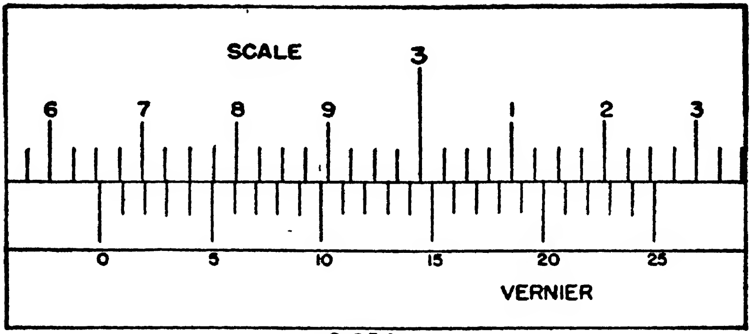
| | Reading on sleeve is between | Reading on thimble between | Vernier line | Complete Reading |
|---|---|---|---|---|
| (a) | .325 and .350 | 22 and 23 | 3 | |
| (b) | .875 and .900 | 8 and 9 | 9 | |
| (c) | .450 and .475 | 16 and 17 | 2 | |
| (d) | .200 and .225 | 11 and 12 | 4 | |
| (e) | .575 and .600 | 2 and 3 | 6 | |
| (f) | .600 and .625 | 24 and 25 | 1 | |

**Vernier Instruments.** The principle of the vernier has been applied to many kinds of instruments. In the foregoing discussion, it was seen how it was employed to increase the limit of measurement of the micrometer. Where the vernier is employed on an instrument as the sole agent for magnifying ordinarily imperceptible differences in length, it is usually known as a vernier caliper, vernier protractor, etc. It consists of a small auxiliary scale having usually one less or more graduations in the same length as the longer true scale. It is evident therefore that if the whole vernier scale contains one more division than the true scale over an equal length, each division on the vernier scale is proportionally

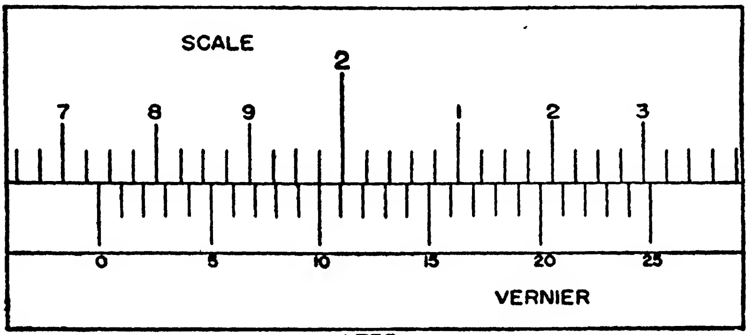



The Vernier Scale

smaller than a corresponding division on the true scale. If 25 divisions on the vernier scale are equal to 24 divisions on the main scale, then each division on the vernier scale is 1/25 of a division smaller than a division on the main scale. If there is an accumulating difference of 1/25 of a division, the effect of going along the vernier scale one division is to subtract 1/25 of a true scale division. By going along the scale 2, 3, 4, 5, etc., divisions of

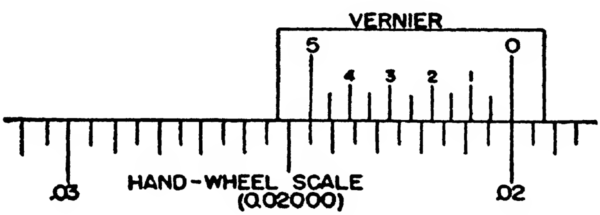


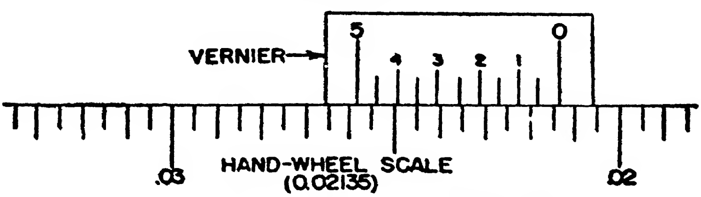

2.654




1.735

FROM A VERNIER CALIPER




FROM HAND-WHEEL OF JIG BORER

VERNIER SCALE READINGS

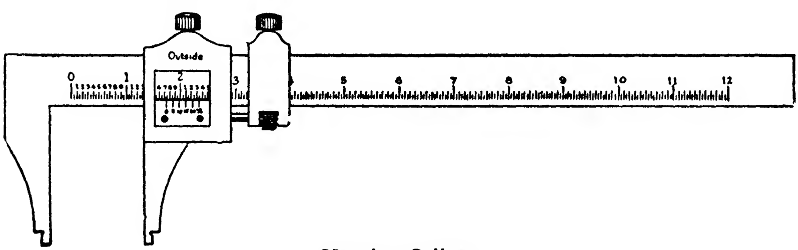
$\frac{2}{25}$, $\frac{3}{25}$, $\frac{4}{25}$, $\frac{5}{25}$, etc., are subtracted from the original setting until the lines coincide. At this point all of the remaining fraction of a division indicated by the "0" on the vernier scale has been absorbed, and the number of the vernier divisions indicates the number of the 25ths this fraction of a division contains.

Vernier scales are not necessarily 25 units long; they may have any number of units. They may have only ten units, as on the vernier scale of the ten-thousandths micrometer. The graduated hand wheels of a machine tool such as a jig borer often employ the vernier scale for the purpose of indicating "tenths" or "half-tenths" of a thousandth of an inch table travel, etc.

**Types of Vernier Instruments.** The vernier scale has been applied to a variety of instruments and tools; for example:

1. Vernier Caliper
2. Vernier Height Gage
3. Vernier Depth Gage
4. Vernier Protractor

The typical vernier caliper consists of an L-shaped frame, the end of which is one of the jaws. On the long arm of the "L" is scribed the true scale, which may be 6, 12, 24, 36, or 48 inches long. The sliding jaw carries a vernier scale on either side. The scale on the front side is for outside measurements, whereas the scale on the back side is for inside measurements. It will be noted in the figure that the tips of the jaws have been formed so as to be capable of making an inside measurement. The thickness of the measuring points is automatically compensated for on the inside scale. The sliding jaw assembly consists of two sections joined by a horizontal screw. By clamping the right-hand section at its approximate location, a final fine adjustment of the movable jaw may be obtained by turning the adjusting nut. The sliding jaw may be clamped in any position with the locking screw shown in the figure on top of the jaw. The jaws of all vernier calipers, except the larger sizes, have two center points which are particularly useful in setting dividers to exact dimensions.

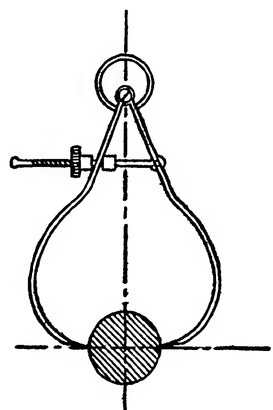



Vernier Caliper

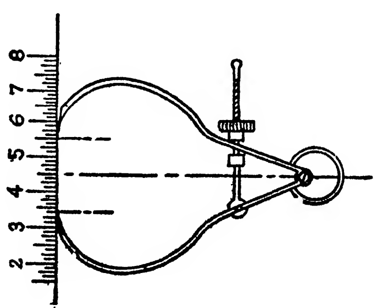
Vernier calipers are made in the standard sizes of 6, 12, 24, 36, and 48 inches, and 150, 300, 600, and 900 millimeters. The length of the jaws will range from 1¼ inches to 3½ inches, and the minimum inside measurement with the smallest caliper is ¼ of an inch or 6 millimeters.

The vernier caliper has a wide range of measurement, and the shape of the measuring anvils and their position with respect to the scale adapts this instrument to certain jobs where a micrometer, for example, could not satisfactorily be applied. It is also capable of being used for both outside and inside measurements—a feature which makes this tool one of the most versatile precision instruments in the shop. However, it does not have the accuracy of a micrometer. In any one inch of its length a vernier caliper should be accurate within .001 of an inch. In any 12 inches it should be accurate within .002, and increase about .001 for every 12 inches thereafter.

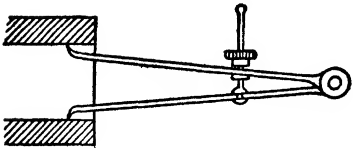
**Other Types of Calipers.** For outside measurements, such as the thickness of a metal plate or the diameter of a cylinder, the *outside* calipers are used as shown herewith. In using these calipers, the instrument must always be kept square with the work to be measured. For inside measurements, such as the inside diameter of a pipe or a tube, the *inside* calipers are used; when using this, the axis of the calipers must line up with the axis of the work, and the tips of the caliper legs must be square with the largest portion of the diameter being measured. In using the micrometer calipers described in the preceding paragraphs the following points should be observed:



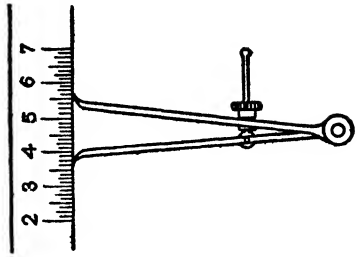
Taking a measurement with an outside calipers.



Transferring the measurement.



Taking a measurement with an inside calipers.




Transferring the measurement.

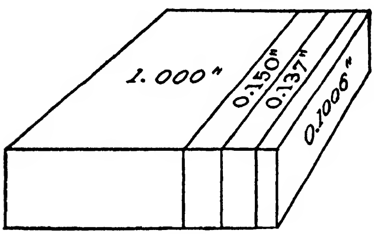
1. Never force the caliper by using too much pressure.
2. Always take the reading while the micrometer is held on the work.
3. Always open the micrometer before removing it from the part measured.
4. Never use the micrometer on a moving part while a machine is running.

**Precision Measurements.** Line-graduated measuring instruments, such as the scale, by which measurements up to .001 inch are taken, are "non-precision instruments," despite the fact that careful, accurate measurements may be made with them. But when dimensions are controlled and reproduced to thousandths of an inch or better, whether by a micrometer or by gage blocks, the measurements are known as *precision measurements.* Such precision measuring instruments themselves are calibrated by means of gage blocks, which are taken as the industrial standards of length. For ordinary shop operations, such as patternmaking, forging, stamping, rough machining, etc., precision measurements are not required. But where tolerances are very small, as for example in the manufacture of watches, clocks, delicate instruments, typewriters, firearms or parts of automotive engines, the use of precision gage blocks is indispensable.

**Gage Blocks.** Gage blocks are rectangular blocks of steel with a measuring surface at each end. They are made of a special alloy steel, heat treated and aged so that internal, molecular stress and strain are at a minimum, thereby decreasing the tendency of the metal to warp or "grow." The surfaces of the blocks are mechanically polished by a special process by which a number of blocks are finished at the same time to identical size. The flat surfaces of the blocks are ground and polished to an extremely high finish resembling that of burnished silver. They are the most accurate pieces of manufactured metal in the world; their errors are generally less than $\frac{8}{1,000,000}$ of an inch per inch of length, and some of them are accurate to within $\frac{2}{1,000,000}$ of an inch. They are probably the

nearest approach of a man-made device to a perfect mathematical plane. Since a rise in temperature of 1° causes the blocks to expand 6/1,000,000 of an inch, they are finished, and subsequently used, in a room kept at a constant temperature of 68°F. Gage blocks usually range in length from .010 of an inch up to 20 inches. They are generally obtainable in sets from 5 to as many as 85 blocks of different lengths. With a large set of over 80 blocks more than 100,000 gages in steps of .0001 of an inch may be made by using various combinations of blocks.

To combine them, the surfaces, having first been thoroughly cleaned, are slid one on the other, with a slight inward pressure; this is sometimes spoken of as "wringing" them together. When placed together in this way they stick with remarkable tenacity; when lifted in the air, blocks that have been wrung together properly have been known to support a weight of somewhat over 200 lb., although the precise reason for this amazing adhesion has never been satisfactorily explained.




Showing how gage blocks are used in combination to make up the measurement of 1.3876

As already mentioned, appropriate gage blocks are put together to secure any desired combination necessary for a particular measurement. For example, if a measurement such as 1.3876″ is desired, the following blocks would be selected: 1.000″; .150″; .137″; .1006″; their sum equals 1.3876″, which is the required measurement. In many cases it will be possible to find several combinations of blocks to give the measurement required. A complete set of blocks may contain the following sizes:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1.000″ | .050″ | .550″ | .101″ | .114″ | .126″ | .138″ | .1001″ |
| 2.000″ | .100″ | .600″ | .102″ | .115″ | .127″ | .139″ | .1002″ |
| 3.000″ | .150″ | .650″ | .103″ | .116″ | .128″ | .140″ | .1003″ |
| 4.000″ | .200″ | .700″ | .104″ | .117″ | .129″ | .141″ | .1004″ |
| | .250″ | .750″ | .105″ | .118″ | .130″ | .142″ | .1005″ |
| .010″ | .300″ | .800″ | .106″ | .119″ | .131″ | .143″ | .1006″ |
| .020″ | .350″ | .850″ | .107″ | .120″ | .132″ | .144″ | .1007″ |
| .030″ | .400″ | .900″ | .108″ | .121″ | .133″ | .145″ | .1008″ |
| .040″ | .450″ | .950″ | .109″ | .122″ | .134″ | .146″ | .1009″ |
| | .500″ | | .110″ | .123″ | .135″ | .147″ | |
| | | | .111″ | .124″ | .136″ | .148″ | |
| | | | .112″ | .125″ | .137″ | .149″ | |
| | | | .113″ | | | | |

*Exercise 14.*

Using the above table of sizes, find appropriate combinations of blocks for each of the following measurements:

| | | | | | |
|---|---|---|---|---|---|
| 1. | .3944 | 5. | .3982 | 9. | 1.8539 |
| 2. | .5532 | 6. | .4338 | 10. | 9.6402 |
| 3. | .4265 | 7. | 3.9061 | 11. | 7.2944 |
| 4. | .666 | 8. | 2.7072 | 12. | 4.0098 |

**Accuracy of Gage Blocks.** The guaranteed accuracy of gage blocks is expressed as ±.000002 of an inch *in an inch.* In other words, a gage block measuring .500 of an inch may vary between .499998 and .500002 and still be acceptable. Furthermore, a block 4.000000 inches long may vary 4 times .000002, or .000008 of an inch, i.e., from 3.999992 to 4.000008 inches and be acceptable. It might be supposed that the accumulated error in a stack of five or six blocks might be considerable; it would be, were it not for the fact that the variations mentioned are distributed according to the laws of probability—some *plus* and some *minus* —so that they counterbalance, and the total error in a stack of blocks rarely exceeds twice that of a single block; frequently it is even *less* than that of a single block.

**Uses of Gage Blocks.** It should also be pointed out that gage blocks are made and used for various classes of work, i.e., various levels of accuracy. Thus a very high grade of blocks, with a range of error of from 5 to 20 millionths of an inch, would be used for inspecting tools, verifying gages, and calibrating various instruments. A second-grade set of blocks, with a range of error of from 20 to 40 millionths of an inch, might be used in layout work,—dies, jigs, fixtures, etc. A third-rate set, with errors ranging from 40 to 100 millionths of an inch, would be suitable for setting up milling, grinding, and drilling machines, or for the inspection of machine parts, etc.

**Standard Sets and Working Sets.** Wherever a considerable amount of precision measuring is required in an industrial plant, it is customary to use primary and secondary standards, that is, a master set of gage blocks, and a working set. The master set is carefully preserved, and is used only to check the accuracy of the working sets used in the shop. The master set is usually sent to the Bureau of Standards at Washington, or returned to the manufacturer, once a year or so for certification; each block is then checked for flatness, parallelism and length, and is certified as varying so-and-so many millionths of an inch.

As the working sets get older they become worn through use and handling. It must be remembered that all gage blocks are extremely delicate; even the natural moisture of the hands contains an acid which may stain the blocks if they are handled too much. Hence as the blocks

wear out, they are progressively used for less important work. When they become so worn that the error is greater than 100 millionths of an inch they are either discarded or chromium-plated and relapped to size. In recent years, gage blocks have been made from carboloy for use in working sets because of the high resistance of carboloy to wear.

## 5. PERCENTAGE

**Meaning of Per Cent.** As we have already seen, a fractional part of any given amount may be expressed either as a common fraction or as a decimal fraction. There is still a third way: by using a per cent. A *per cent* is simply a decimal fraction written without the decimal point, with the (%) sign used instead of the decimal point to indicate the fact that the denominator is "hundredths"; thus

$$\begin{aligned} .28 &= 28\% \\ .125 &= 12.5\% = 12\tfrac{1}{2}\% \\ .003 &= .3\% = \tfrac{3}{10}\% \\ .0425 &= 4.25\% = 4\tfrac{1}{4}\% \end{aligned}$$

Per cents may be added, subtracted, multiplied, etc., just as other numbers having similar units or denominations; thus

$$\begin{aligned} 15\% + 3\% + 10\% &= 28\% \\ 100\% - 16\% &= 84\% \\ 6 \times 3\tfrac{1}{2}\% &= 21\% \\ 24\% \div 3 &= 8\% \end{aligned}$$

**Finding a Per Cent of a Number.** The commonest problem involving per cents is that of finding a given per cent of a given number. The given number is called the *base;* the per cent required is called the *rate;* and the result of finding the per cent (or taking the rate) is called the *percentage.* Thus, in finding 20% of 750, we say:

$$\begin{aligned} 20\% \text{ of } 750 &= ? \\ \text{or} \quad .20 \times 750 &= 150; \end{aligned}$$

here 750=base, .20=rate per cent, and 150=percentage. Finding a per cent of a number is therefore seen to be a simple matter of multiplying a number by a decimal, i.e., using the formula

$$\text{Percentage} = \text{Base} \times \text{Rate}$$
$$\text{or,} \quad P = B \times R$$

EXAMPLE 1: The cost of material for a job is estimated at $48; an additional 15% is allowed for miscellaneous expenses. Find (a) the amount of the allowance, and (b) the total estimated cost.

SOLUTION: (a) \$48×.15=\$7.20, *Ans.*

(b) \$48+\$7.20=\$55.20, *Ans.*

EXAMPLE 2: A specimen of Monel metal consists of 68% of nickel, 28% of copper, and the remainder of small amounts of other metals. Find (a) the number of pounds of nickel and copper in a piece of Monel metal weighing 18½ lb.; (b) the number of pounds of material other than nickel and copper in this specimen.

SOLUTION: (a) 18.5×.68=12.58 lb. nickel, *Ans.*
18.5×.28= 5.18 lb. copper, *Ans.*
(b) 68%+28%=96%
100%—96%= 4%, other metals
18.5×.04 =.74 lb., *Ans.*

*Exercise 15.*

1. If ice is 91.7% as heavy as water, and water weighs 62.4 lb. per cu. ft., find the weight of a cubic foot of ice.
2. When tested, a gasoline engine actually gave 86% of its rated horsepower. If the engine was rated at 110 H.P., what was the actual horsepower delivered by the engine?
3. A motor is running at 2600 revolutions per minute. If the speed of the motor is increased by 6½%, how many r.p.m. will it then make?
4. If the loss in power due to friction in a certain device is 28%, what amount of power will this device transmit when supplied with 125 horsepower?
5. A machine shop casting weighs 60 lb. Due to an error in dimensioning, it is necessary to remove 12½% of the casting by machine. How many pounds of metal must the machinist remove?
6. The employees in a shop are to receive a wage increase of 12%. If junior mechanics have been getting \$10.75 a day and helpers \$7.80, what is the daily wage rate of each after the raise goes into effect?
7. An alloy used for bearing metal contains 14% tin, 27% antimony and 59% lead; how much of each of these metals is required to make 150 lb. of bearing metal?
8. A pattern weighs ¼ as much as the casting to be made from the pattern. If the casting weighs 110 lb., what is the weight of the pattern?
9. An inexperienced operator turned out 250 pieces of work on a stamping machine When inspected, it was found that 2½% of them had to be rejected as imperfect. How many pieces were rejected?
10. In making a certain piece to measurement, an allowance of 1½% either way is permitted. If the dimension called for is 4.8 in., what are the "outside limits" permissible?

11. In mixing a batch of concrete, about 15% of the weight is cement, 30% is sand, and 55% is gravel. If the dry mixture has a total weight of 1450 lb., how many pounds of each are used?

**Changing a Per Cent to a Fraction.** It is sometimes convenient to change a per cent into an equivalent common fraction. To do this, the per cent is first expressed as a decimal fraction, which in turn is then reduced to lowest terms.

EXAMPLE 1: Change 28% to a common fraction.

SOLUTION: $28\% = .28 = {}^{28}\!/\!_{100}$

$^{28}\!/\!_{100} = {}^{14}\!/\!_{50} = {}^{7}\!/\!_{25}$, *Ans.*

EXAMPLE 2: Express $18\frac{3}{4}\%$ as a common fraction.

SOLUTION: $18\frac{3}{4}\% = .18\frac{3}{4} = .1875 = \dfrac{1875}{10{,}000} = {}^{75}\!/\!_{400} = {}^{3}\!/\!_{16}$, *Ans.*

**Changing a Common Fraction to a Per Cent.** In a somewhat similar way, the above process may be reversed, and any common fraction can in turn be expressed as a per cent. Thus the numerator is first divided by the denominator, the quotient being written as a decimal; the decimal is then converted to a per cent by moving the decimal point two places to the right and annexing the % sign.

EXAMPLE 1: Change ⅜ to the per cent form

SOLUTION: $\frac{3}{8} = 3 \div 8$

$$8\overline{)3.000}$$
$$.375$$

$.375 = 37.5\%$, *Ans.*

EXAMPLE 2: Express $^{13}\!/\!_{17}$ as a per cent.

SOLUTION:

```
       .7647+
17)13.000
    11 9
     1 10
     1 02
        80
        68
       120
       119
```

or, $.7647 = 76.47\%+$, *Ans.*

EXAMPLE 3: Change $\frac{6}{5}$ to a per cent.

SOLUTION: $6 \div 5 = 1.2 = \frac{120}{100} = 120\%$, *Ans.*

**Determining the Rate Per Cent.** Another common problem arising in connection with the use of per cents is that of determining the rate per cent, i.e., finding what per cent one number is of another. It is precisely the same problem as that explained in the preceding paragraph. Its relation to percentage will be seen at once from the following:

$$Percentage = Base \times Rate$$

If $\quad P = B \times R,$

then $R = \frac{P}{B}$

or $\quad Rate = Percentage \div Base.$

EXAMPLE 1: What per cent of \$600 is \$39?

SOLUTION: $\frac{\$39}{\$600} = \frac{39}{600} = \frac{13}{200} = .065 = 6\frac{1}{2}\%$, *Ans.*

EXAMPLE 2: By changing the plan of a pattern a saving of $5\frac{1}{4}$ lb. is made in a casting originally weighing 46 lb. What is the per cent of weight thus saved?

SOLUTION:
$$\frac{5\frac{1}{4}}{46} = \begin{array}{r} .114+ \\ 46\overline{)5.25} \\ \underline{4\,6} \\ 65 \\ \underline{46} \\ 190 \\ \underline{184} \end{array}$$

Saving $= .114+ = 11.4\%+$, *Ans.*

*Exercise 16.*

1. An automatic production machine turns out 36 pieces of work per hour. After certain adjustments had been made, the machine turned out 42 pieces per hour. What is the per cent of increase in the production rate?
2. A shop hand receiving 80¢ an hour is given an increase in wages. If he now receives 92¢ an hour, by what per cent was his wage rate increased?

3. Before a bronze casting was machined it weighed 31¼ lb.; after the machining operations had been performed it weighed 27½ lb. What was the per cent of reduction in weight?
4. A steam pressure of 175 lb. per sq. in. is increased to 220 lb. per sq. in. What is the per cent of increase?
5. If the price of gasoline is increased from 16½¢ a gallon to 17¼¢, what per cent of increase is this?
6. A power saw uses a "cutting compound" made by mixing 5 quarts of lard oil with water enough to fill a 10-gallon tank. What per cent of the compound is oil?
7. A beam is expected to support a maximum load of 40 lb. per linear foot. If it is designed to withstand a load of 56 lb. per ft., what factor of safety was allowed?
8. In making a pattern, a designer allows 3/16 in. per foot for shrinkage. What per cent is this?
9. A bottle contains 250 gm. of potassium chloride "analyzed reagent." The label states that it contains 0.083 gm. of magnesium chloride impurity. What per cent is this?
10. A carpenter added 1¼ pints of alcohol to 2½ quarts of shellac. By what per cent did he "thin" the shellac?
11. In a printing of 2500 leaflets the press operator spoiled 45 copies. What was the per cent of spoilage?
12. By tuning up a Diesel engine an operator saves an average of 15 gal. of fuel per day. If the average consumption of fuel had been 175 gal. per day, what was the per cent of saving in fuel?

**Finding the Base.** A less frequently occurring problem is the following: having given the percentage and the rate per cent, what was the original base? When expressed by means of the formula, the method of answering this type of question may readily be seen; thus

$$\text{since } P=B\times R,$$
$$\text{then } B=P\div R,$$

or the base is found by dividing the percentage by the rate.

EXAMPLE 1: A factory "let out" 240 of its employees. If this meant that 15% of its employees were laid off, how many were originally employed?

SOLUTION: $15\%=240$
$1\%=240\div 15=16$
$100\%=100\times 16=1600$, *Ans.*

EXAMPLE 2: By increasing the amount of "filler" in a certain grade of paper stock the weight of the paper was increased by 10%. If the stock now weighs 16½ lb., what did it weigh originally?

SOLUTION: $100\%+10\%=110\%$
$110\%=16.5$ lb.
$1\%=16.5\div 110=.15$ lb.
$100\%=100\times .15=15$ lb., *Ans.*

*Exercise 17.*

1. In normal times a shop produces a certain number of finished pieces per day. When production is stepped up 25% by working overtime, 39 additional pieces per day are produced. What is the daily production under normal conditions? when overtime work is done?
2. A certain material shrank, on account of moisture, 4% in size. If it now measures 21.6″, what was its original size?
3. The inventory clerk of a factory has on hand 300 ft. of round bars of a certain size. If this is 30% of the average stock of that size, how much of it is usually kept on hand?
4. The weight of zinc in a casting made of Lumen metal is 74.8 lb. If Lumen metal consists of 5% aluminum, 10% copper, and the rest zinc, find the weight of the casting.
5. The overhead in a manufacturing plant is 24% of the value of the goods produced. If in a certain month the overhead expense amounted to $38,400, what was the value of the goods produced?
6. A foreman's weekly wage is increased by 12%. If his raise amounts to $5.82 per week, what was his original weekly wage?

**Practical Uses of Percentage.** Many practical applications of percentage problems arise in the trades, especially in connection with the business aspects of industrial practice. Thus profit is expressed as a certain per cent of the volume of sales, or sometimes as a per cent of the cost; so also, are labor costs, overhead, and cost of materials expressed as per cents. Other items frequently expressed in per cents are the cost of maintenance of physical plant and equipment; the cost of repairs and replacements; taxes; insurance, such as fire insurance, flywheel insurance, boiler explosion, plate-glass insurance, etc., the premiums on accident policies, workmen's compensation, unemployment insurance, pensions, old-age security benefits and the like; all these are usually figured in terms of percentages.

**Depreciation.** All equipment, such as machinery, tools, fixtures, trucks, even buildings, decrease in value as time goes on. This decrease in value is known as *depreciation*. It is usually due to the actual wearing out of the equipment so that it is no longer serviceable, although it sometimes becomes necessary to discard equipment even before it is worn out completely because of new inventions or improved styles. Frequently the equipment to be discarded has a certain junk value, or scrap value, when it is disposed of. The number of years that the equipment remains

in use is called its estimated or "useful life". The amount of annual depreciation can be computed in several ways. One of the simplest and commonest methods is to suppose that it depreciates in value in *equal amounts* each year of its life. This is not actually the case with some types of equipment. However, when this method is used, the computation is as follows:

$$\textit{Annual depreciation} = \frac{C - S}{n}$$

where $C$=original cost, $S$=scrap value, and $n$=number of years of estimated life. On this basis (called the "constant-value" method), the *rate of depreciation* is given by:

$$\textit{Rate of depreciation} = \frac{\textit{Annual depreciation}}{\textit{Original cost}}$$

**Example:** A power drill worth $875 when new has an estimated life of 15 years, and its scrap value is $50. Using the constant-value method, find
(a) the annual depreciation charge, and
(b) the annual rate of depreciation.

**Solution:** (a) Amount of annual depreciation $= \frac{\$875 - \$50}{15} = \frac{\$825}{15} = \$55$, *Ans*

(b) Rate of annual depreciation $= \frac{\$55}{\$875} = .0629 = 6.29\%$, *Ans.*

**Commercial Discount.** When material or equipment is purchased, it is usually subject to *discount*. This means that a certain per cent of the quoted price is allowed, either for immediate cash payment, or because of a trade allowance on the list price, or because it is bought in large quantity. The following examples will illustrate such discounts.

**Example 1:** A bill for lumber amounted to $176; a discount of 2½% for cash was allowed. If prompt payment was made, what did the lumber actually cost the purchaser?

**Solution:** $\$176 \times .025 = \$4.40$, discount
$\$176 - \$4.40 = \$171.60$, net cost, *Ans.*

**Example 2:** A wrench listed in the catalog at $2.75 is subject to a discount of 33⅓%. What is the net cost?

**Solution:** $\$2.75 \times 33\frac{1}{3}\% = \$.916 = \$.92$, discount
$\$2.75 - \$.92 = \$1.83$, net cost, *Ans.*

EXAMPLE 3: A supply house lists a rotary pump at \$75, subject to a discount of 20% and 10%. Find the net price.

SOLUTION:
$75×20%=$15, first discount
$75—$15 =$60, first "net price"
$60×10%=$ 6, second discount
$60—$6 =$54, net price, *Ans.*

NOTE: Either discount can be computed first; the result will be the same. The two discounts *cannot be added,* however; "20% and 10%" is *not equivalent* to a 30% discount. This is the short cut, if you want one:

100%—20%=80%
100%—10%=90%
(.9)×(.8)=.72=72%
$75×.72=$54, *Ans.*

*Exercise 18.*

1. Brass fittings are offered by a manufacturer at \$7.20 a hundred. At a discount of 25%, what is the cost of 200 fittings?
2. Micrometers are quoted at \$32 a doz., less 25% and 20%. What is the net cost of one micrometer?
3. A wood plane was purchased at a net cost of \$3.80. If the catalog price was \$4.75, what per cent of discount was offered?
4. A lathe costing \$1200 has an estimated life of 15 years. If it has a scrap value of \$150, what is the annual depreciation charge?
5. If a piece of equipment costing \$240 has no scrap value and has a useful life of 8 years, what is the annual rate of depreciation?
6. A factory building cost \$40,000 to erect. If the depreciation is figured at 2% of the original cost each year, what is the "book value" of the building after 22 years?
7. A solution of a solid in water contains 24% of the solid by weight. How much of this solid is dissolved in 15 oz. of the solution?
8. As calculated, the speed of a pulley is 300 revolutions per minute. When checked with a tachometer, however, it actually made only 280 r.p.m., due to belt slippage. What is the per cent of slippage?
9. Ordinary air contains about 19.5% of oxygen by volume. How many cubic feet of oxygen are there in 2000 cu. ft. of air?
10. Because of a leaky valve, 2½ quarts of oil were lost out of every 20 gallons. What per cent was lost?
11. A factory regularly employing 320 men increases its force by 15%. How many men are now working in this factory?
12. The diameter of a rod when measured by a micrometer is found to be 1.23 in. If the blueprint called for a diameter of 1.18 in., what is the per cent of error?

13. If the working hours are increased from 40 hours per week to a 44-hour week, what is the increase in a weekly pay roll amounting to $5600, if the same hourly wage rates are maintained?
14. An upholsterer allows 10% extra on the cost of material for nails, thread, glue, sandpaper, etc. If the wood and fabric for an upholstered stool come to $4.38, what is the total estimated cost of material for 2 doz. such stools?
15. When making a bid on an installation job, a contractor allows 30% of the estimated cost for additional "overhead." If he offers a bid of $156, what did he estimate the cost to be?

## 6. RATIO AND PROPORTION

**Ratio as a Comparison.** A *ratio* is a device for comparing two quantities of the same kind. For example, if two strips of metal are 8 in. and 10 in. long, respectively, we could say that the second is 2 in. longer than the first, or 25% longer than the first. This is a *difference* method of comparing them; telling how much more or less. Another way of comparing them would be to say that one is $\frac{4}{5}$ as long as the other, or the second is $\frac{5}{4}$, i.e., $1\frac{1}{4}$, times as long as the first. This is the *ratio* method of comparison: telling how many times as much. We say that the lengths of the boards are "in the ratio of 4 to 5, or 5 to 4," which may be written as 4:5, or 5:4. A ratio, then, is simply a *fraction* which gives the comparison at a glance; the above ratio might also be written as $\frac{4}{5}$ or $\frac{5}{4}$ instead of "4:5" or "5:4"; in fact, the colon (:) is really an abbreviation for "÷" with the horizontal line omitted. Notice that a ratio is independent of the units of measure; i.e., the two lengths mentioned above are in the ratio of 4:5 whether we express them in inches, feet, or yards. The units "cancel out," and the ratio remains 4:5. When comparing two quantities by the ratio method, however, care should be taken that the numbers to be compared are always expressed in the same units of measure to begin with.

EXAMPLE 1: A spindle is 3″ high and ¾″ in diameter. What is the ratio of its diameter to its height?

SOLUTION: $\frac{3}{4} \div 3 = \frac{3}{4} \times \frac{1}{3} = \frac{1}{4} = 1:4$, *Ans.*

EXAMPLE 2: A rectangular sheet of tin measures 12′6″ in length by 8′4″ in width. Find the ratio of the length to the width.

SOLUTION: $12'6'' = 12\frac{1}{2}$ ft.
$8'4'' = 8\frac{1}{3}$ ft.
$12\frac{1}{2} \div 8\frac{1}{3} = \frac{25}{2} \times \frac{3}{25} = \frac{3}{2} = 3:2$, *Ans.*

*Exercise 19.*

1. Two ladders are 12 ft. and 18 ft. long. What is the ratio of their lengths?
2. What is the ratio of the lengths of a 6″-pocket rule and a yardstick?
3. Two near-by office buildings are 24 stories and 36 stories high. Assuming that the "stories" in each building are the same height, what is the ratio of the heights of the buildings?
4. A certain style "legal size" envelope measures $4'' \times 9\frac{1}{2}''$. What is the ratio of its width to its length?
5. A photographic print is $3\frac{1}{4}'' \times 4\frac{1}{4}''$. What is the ratio of its dimensions?
6. A rectangle is said to have the most pleasing appearance when the ratio of its width to its length is 0.7. According to this standard, what should be the width of a rectangular placard that is 25 in. long?
7. A boy is 3 ft. 9 in. tall, and his father stands 5 ft. 9 in. Find the ratio of their heights.
8. A drawing of a flower in a biology textbook is $5\frac{1}{4}$ in. high. If the caption under the drawing reads "$\frac{3}{4}$ actual size," what is the actual height of the flower?
9. The micro-photograph of a textile fibre is 2.4 cm. long. If the magnification is 1:60, how long is the actual specimen?
10. A mechanic constructed a miniature model of a machine part which was actually 3 ft. 6″ long. If he used a scale of "1 inch$=\frac{1}{2}$ foot," how long did he make the model?

**Using Ratios.** Ratios are very useful, and can be employed in many ways, as the following illustrative problems will show.

EXAMPLE 1: A board is 16 ft. long; if it is to be divided into two pieces in the ratio of 3:5, how long should each piece be?

SOLUTION: $3+5=8$

one piece $=\frac{3}{8}$ of entire length, or 6 ft.
other piece$=\frac{5}{8}$ of entire length, or 10 ft., *Ans.*

EXAMPLE 2: Muntz metal consists of 6 parts of copper and 4 parts of zinc by weight. How many pounds of each metal are there in a block of Muntz metal weighing 72 lb.?

SOLUTION: $6+4=10$
Ratio of copper to Muntz$=6:10=.6$
Ratio of zinc to Muntz $=4:10=.4$
$72 \times .6=43.2$ lb. copper,
$72 \times .4=28.8$ lb. zinc, *Ans.*

**Example 3:** The ratio of the diagonal of a square to the side of the square is 1.4. Find (a) the diagonal of a square whose side is 30 inches; (b) the side of a square whose diagonal is 28 inches.

**Solution:** A ratio of 1.4 is the same as 14:10, or (1.4):(1).

(a) diagonal $= 1.4 \times$ side

$= 1.4 \times 30 = 42$ in., *Ans.*

(b) side $=$ diagonal $\div 1.4$

$= 28 \div 1.4 = 20$ in., *Ans.*

Since a ratio is always a fraction, ratios are frequently expressed as per cents. The specific gravity of a substance is the ratio of its weight to the weight of an equal volume of water. Thus ether, being only about 70% as heavy as water, has a specific gravity of 0.7; ice, 0.92; air, 0.0013; aluminum, 2.6; lead, 11.37; etc.

*Exercise 20.*

1. One inch is equivalent to 2.5 cm. What is the ratio of an inch to a centimeter? of a centimeter to an inch?
2. One liter is equivalent to 1.06 quarts. What is the ratio of a quart to a liter?
3. Divide a 42″-rod into two pieces in the ratio of 5:7.
4. The angles of a triangle are in the ratio of 1:2:3. If the sum of the three angles equals 180°, how large is each angle?
5. One quart equals approximately .95 liters. How many liters of acid are there in a 5-gallon acid carboy?
6. The ratio of the altitude of an equilateral triangle to its side is .866. What is the altitude of such a triangle if its side is 20 inches? What is the length of the side if the altitude is 2.598 inches?
7. The smaller of two connected pulleys makes 180 revolutions per minute while the larger one makes 45 revolutions. What is the ratio of their speeds? If the smaller one is speeded up to 220 r.p.m., what will be the speed of the larger, assuming the same speed ratio?
8. Aluminum metal expands .000013 of its length per Fahrenheit degree rise in temperature. If the original length of an aluminum bar is 200 cm., what is its length when raised 100°F?
9. Monel metal consists of 68½% of nickel, 1½% of iron, and the rest, copper. How many pounds of copper are there in a Monel metal casting weighing 60 lb.?
10. A ton of ready-mix concrete consists of cement, sand and gravel in the ratio of 1½:3½:5. How many pounds of each ingredient are there in the mixture?
11. If a sample of petroleum weighs 55 lb. per cu. ft., and water weighs 62.5 lb. per cu. ft., find the specific gravity of the petroleum.

12. If the specific gravity of ice is 0.92, what is the weight of 8 cu. ft. of ice, assuming that water weighs 62.5 lb. per cu. ft.
13. Brazing metal is an alloy made up of 20% zinc and 80% copper. What is the ratio of zinc to copper?
14. German silver (white metal) consists of 2 parts zinc, 3 parts nickel, and 5 parts copper. Find (a) the ratio of zinc to copper; (b) of copper to nickel; (c) what per cent of the alloy is nickel?
15. A commonly used mixture for concrete is made up of 1 part of cement, 2½ parts of sand, and 4 parts of stone. Find (a) the ratio of sand to stone; (b) the ratio of cement to sand; (c) what per cent of the concrete mixture is sand?

**Scale Drawings.** In representing distances on a map or dimensions on a plan or blueprint, it is necessary to use a scale, or to "scale down" the quantities, all in the same ratio. Thus on a given map, an inch might represent 300 miles, in which case two cities located 2½ inches apart on the map would actually be 750 miles distant from each other. Or the floor plan of a house might be drawn to a scale of 1″=10 ft.; in that case a room which on the plan is ¾″ wide is actually 7½ ft. wide, and a room 18 ft. long would be represented by a line 1.8 in. long.

EXAMPLE 1: A catalog picture of a machine part is labeled as being "⅖ actual size." If the length of the part in the picture is 3.8 in., what is its actual length?

SOLUTION: $3.8 \times 5/2 = 9.5$ in., *Ans.*

EXAMPLE 2: The working model of a machine is to be on a scale of 1:50. If a connecting piece of this machine is actually 16′8″ long, how long should the corresponding piece of the model be made?

SOLUTION: 16′8″=200″
200″×1/50=4″, *Ans.*

EXAMPLE 3: On a blueprint the scale used is 1″=10′. Find (a) the actual size of a distance of 2¼″ on the blueprint; (b) how long on the blueprint an actual distance of 36 ft. ought to be.

SOLUTION: 1″=10 ft.
(a) 2¼″=10×2¼=22.5 ft., *Ans.*
(b) 1 ft.=1/10 in.
36 ft.=36×1/10=3.6 in., *Ans.*

*Exercise 21.*

1. The dimensions of the top of a rectangular workbench are 4′ by 8½′. What should be its dimensions on a scale drawing, if the scale is ¼″=1′?

2. A living room is $20' \times 14'$. On the architect's plan, what are its dimensions, if the scale used is $\frac{1}{8}''=1'$?
3. If the scale used in the following is $1''=1'$, fill in the missing values:

| *Actual Length* | 2′ | 5′ | 8′ | 20′ | | 6½′ |
|---|---|---|---|---|---|---|
| *Scale Length* | 2″ | 5″ | | | 15″ | |

4. If the scale used in the following is $\frac{1}{2}''=10'$, fill in the missing values:

| *Actual Length* | | 20′ | 5′ | 24′ | | |
|---|---|---|---|---|---|---|
| *Scale Length* | ⅛″ | | | | 2″ | 3⅝″ |

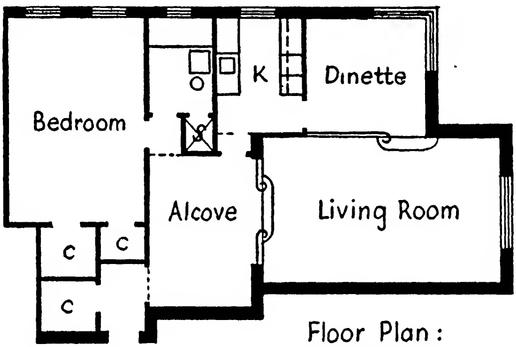
5. Complete the following table:

| *Scale Used* | *Actual length* | *Scale length* | *Actual length* | *Scale length* |
|---|---|---|---|---|
| (a) 1″=20 ft. | 35 ft. | ? | ? | 3¼″ |
| (b) 1″= 6″ | 1½ ft. | ? | ? | 10″ |
| (c) ⅛″= 1 ft. | 12 ft. | ? | ? | 2⅜″ |
| (d) ½″= 1 mile | 25 mi. | ? | ? | 5¾″ |
| (e) ¼″=10 ft. | 45 ft. | ? | ? | 4½″ |

6. The floor space of a storage bin is 14 ft. by 24½ ft. Using a scale of $1''=1'$, what are the dimensions of the floor space on the architect's plan?
7. The dimensions of a metal plate for a machine are 6½ ft.$\times$9 ft. What should its dimensions be on a blueprint, if the scale used is $\frac{1}{2}''=1'$?
8. The scale of miles on a map is $1''=150$ mi. How far apart on the map are two cities that are actually 1225 miles from each other?
9. On the plan of an apartment house a bedroom is shown, measuring 3½″ by 5¾″. If the floor plan is drawn to a scale of $\frac{1}{4}''=1'$, what are the actual dimensions of the room?
10. The detailed plan of a working model is represented on a draftsman's drawing by a scale of $1''=6''$. What are the dimensions of a rectangular part measuring 4½″$\times$6¼″ on the drawing?
11. A surveyor's map is drawn on a scale of $1''=10$ feet. How far is it actually from one point to another that is 3¾ inches from it on the map?
12. On the layout of a camp site $1''=0.5$ mile. It is 1¼ miles from the mess hall to a certain cottage. How far apart are these two places on the diagram?
13. Complete the blank spaces:

| | | | | | | |
|---|---|---|---|---|---|---|
| *Length of object* | 1′ | 5′ | 6″ | | 5″ | 4 yd. |
| *Length of drawing* | 1″ | ½″ | | 3′ | | 6″ |
| *Scale used* | | | ½ | ¼ | 1′=6″ | |

14. A carpenter is using a blueprint with a scale of ½″=1′. What are the dimensions of a door that is 1¾″×3½″ on the blueprint?
15. Find the dimensions of (a) the living room, (b) the dinette, (c) the bedroom, and (d) the alcove.




Floor Plan:
3½ Room Apartment
Scale 1/16″ = 1′ 0″

**Proportion.** The word "proportion" is one which is often used carelessly, or with only a vague notion of what is really meant. Strictly speaking, when we compare two quantities, we cannot speak of their "proportion"; we can refer to the ratio between them, or what part or what per cent one is of the other. When we speak of a proportion, we have in mind *four* quantities, and are comparing them *in pairs,* as ratios. In other words, if two ratios are equal to each other, they are said to form a proportion. Thus, 3:5=12:20 is a proportion. Putting it another way, if the ratio between any two quantities is numerically equal to the ratio between two other quantities, then the four quantities are in proportion. For example, a nickel bears the same ratio to a dime that a half-dollar does to a dollar, since

$$5/10=50/100, \qquad \text{or } 1/2=1/2$$

Notice that in the first case, all four quantities are expressed in terms of the same units, *viz.,* cents, although the units don't appear; in the second case, the two ratios are reduced to lowest terms to show their equality. Or again, if a man 6 ft. tall casts a shadow 8 ft., then a pole 18 ft. high will cast a shadow 24 ft. long; or, 6:8=18:24.

$$6/8=18/24 \qquad 3/4=3/4$$

In other words, a proportion is simply an equation stating that two fractions are equal. For example,

$$\text{if } 2:3=10:15, \qquad \text{then } \tfrac{2}{3}=\tfrac{10}{15},$$

or $2\times15=3\times10$, which might be called "cross-multiplying" the four *terms* of the proportion. If only three of the four terms of a proportion are known, the remaining term can easily be found by the principle of "cross-multiplication," as shown below.

**Examples:**

1. If $\frac{2}{3}=8/n$, find the missing number "*n*."
   Multiplying "crisscross,"
   $2\times n=8\times3$
   $2n=24$
   $n=12$
   *Check:* $\frac{2}{3}=\frac{8}{12}$

2. If $\frac{4}{5}=n/20$, find the missing quantity "*n*."
   Cross-multiplying:
   $4\times20=5\times n$
   $5n=80$
   $n=16$
   *Check:* $\frac{4}{5}=\frac{16}{20}$

3. If $8/n=\frac{5}{8}$, find $n$.
   Multiplying:
   $3\times8=5\times n$
   $5n=24$
   $n=\frac{24}{5}=4\frac{4}{5}$
   *Check:* $3\div4\frac{4}{5}=\frac{5}{8}$

4. If $n/4=\frac{2}{7}$, find $n$.
   $7\times n=2\times4$
   $7n=8$
   $n=\frac{8}{7}=1\frac{1}{7}$
   *Check:* $1\frac{1}{7}\div4=\frac{2}{7}$

*Exercise 22.*

Find the missing term in each of the following proportions:

| | | | |
|---|---|---|---|
| 1. $\frac{3}{4}=15/n$ | 4. $\frac{5}{6}=n/24$ | 7. $6/n=\frac{2}{3}$ | 10. $n/40=\frac{3}{5}$ |
| 2. $\frac{2}{5}=16/n$ | 5. $\frac{3}{8}=n/32$ | 8. $7/n=\frac{1}{6}$ | 11. $n/180=\frac{24}{720}$ |
| 3. $\frac{2}{3}=14/n$ | 6. $\frac{9}{4}=n/12$ | 9. $n/16=\frac{3}{8}$ | 12. $88/n=3\frac{1}{7}$ |

**Direct Proportion.** A proportion in which the ratios vary in the same order is called a *direct proportion*. For example, the volume of a gas (under constant pressure) *varies directly* with the temperature: as the temperature increases, the volume increases; as the temperature decreases, the volume decreases. This may be expressed mathematically as follows:

$$\frac{V_1}{V_2}=\frac{T_1}{T_2};$$

note that the "subscripts" of the letters are in the same order.

**Example 1:** If 8 bolts weigh 10 oz., how much will 48 similar bolts weigh?

**Solution:** Let $x$ represent the required weight.

$$8/48 = 10/x$$
$$8x = 10 \times 48$$
$$x = \frac{10 \times 48}{8} = 60 \text{ oz., } \textit{Ans.}$$

**Example 2:** The elongation of a certain spring varies directly with the weight applied. If a weight of 48 oz. causes an elongation of 2 inches, (a) what will the elongation be when a weight of 60 oz. is applied? (b) what weight will produce an elongation of 1½ inches?

**Solution:**

$$\frac{e_1}{e_2} = \frac{w_1}{w_2}$$

(a) $2/x = 48/60$
$$48x = 2 \times 60$$
$$x = \frac{2 \times 60}{48} = 2\tfrac{1}{2} \text{ in., } \textit{Ans.}$$

(b) $\dfrac{2}{1\frac{1}{2}} = \dfrac{48}{x}$
$$2x = \tfrac{3}{2} \times 48$$
$$x = \frac{\frac{3}{2} \times 48}{2} = 36 \text{ oz., } \textit{Ans.}$$

## *Exercise 23.*

1. If 100 pages of a book measure ⅜″ in thickness, what will be the thickness of a book of 480 pages of the same quality of paper?
2. If 7½ gallons of paint cost $10.50, what will 40 gallons of paint cost?
3. Eight stamping machines turn out 560 pieces of work in one hour. How many pieces will 5 of these machines turn out in an hour and a half?
4. If a bomber flies 840 miles in 3 hours, how far will it fly in 8½ hours at the same rate?
5. If the electrical resistance of 250 ft. of a certain wire is 150 ohms, how many ohms resistance will 875 ft. of the same wire have?
6. Metal castings are often sold by the pound. If a casting weighing 240 lb. costs $12.80, what is the weight of a similar casting that costs $41.60?

7. A section of a steel girder 18 ft. in length weighs 450 lb. How long is another section of the same girder if it weighs 1050 lb.? What is the weight of a piece 10 ft. long?
8. The volume of a gas under constant pressure varies directly as the absolute temperature. If $V=420$ when $T=225$, what is the value of $V$ when $T=175$? For what value of $T$ will $V$ equal 1200?
9. At 75 lb. pressure per sq. in., a certain exhaust pipe discharges 270 cu. ft. of gas per minute. Assuming direct variation, how many cu. ft. are discharged at 80 lb. pressure? What pressure is required to discharge 450 cu. ft. per minute?
10. The distance traveled by sound varies directly as the time required to hear the sound. A storm is 1½ miles away, and the sound of the thunder reaches an observer 7.2 seconds after the lightning flash is seen. Some time later the thunder is heard 4.5 seconds after the flash. How much nearer is the storm the second time?

**Inverse Proportion.** A proportion in which the ratios vary in the opposite order is called an *inverse proportion.* For example, the volume of a gas (under constant temperature) *varies inversely* as the pressure: as the pressure increases, the volume *decreases,* and as the pressure decreases, the volume *increases.* This may be expressed mathematically as follows:

$$\frac{V_1}{V_2}=\frac{P_2}{P_1};$$

note that in this case the subscripts of the letters are in *reverse* order.

EXAMPLE: The current ($C$) in an electric circuit varies inversely as the resistance ($R$). If $C=2$ amperes when $R=55$ ohms, find (a) the current when $R=220$; (b) find the resistance $R$ when $C=5$ amperes.

SOLUTION: $\dfrac{C_1}{C_2}=\dfrac{R_2}{R_1}$

(a) $\dfrac{2}{C_2}=\dfrac{220}{55}$

$220\ C_2=2\times55$

$C_2=110\div220=\frac{1}{2}$ ampere, *Ans.*

(b) $\dfrac{2}{5}=\dfrac{R_2}{55}$

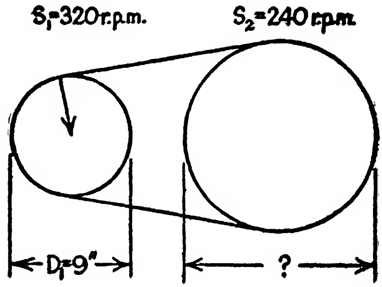
$5R_2=2\times55$

$R_2=110\div5=22$ ohms, *Ans.*

Inverse proportion is well illustrated by the relation of the diameters of pulleys and gears. Whenever two pulleys having different diameters are

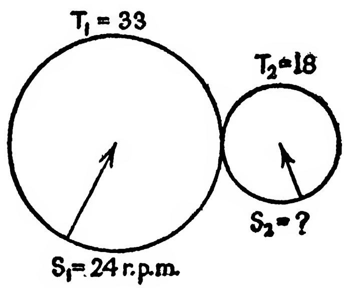
connected, the smaller pulley always rotates more times than the larger. Likewise, when two gears with different diameters are in mesh, the smaller (having the lesser number of teeth) turns more rapidly than the larger one.

EXAMPLE 1: Two pulleys are connected with a belt; the smaller, having a diameter of 9″, makes 320 revolutions per minute, while the larger makes 240 r.p.m. Find the diameter of the larger pulley.




SOLUTION:
$$\frac{D_1}{D_2}=\frac{S_2}{S_1}$$
$$\frac{9}{D_2}=\frac{240}{320}$$
$$240\ D_2=9\times 320$$
$$D_2=16'', \textit{ Ans.}$$

EXAMPLE 2: Two gears in mesh have 33 teeth and 18 teeth. When the larger makes 24 r.p.m., how many r.p.m. does the smaller gear make?

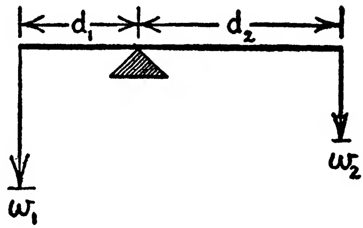



SOLUTION:
$$\frac{T_1}{T_2}=\frac{S_2}{S_1}$$
$$\frac{33}{18}=\frac{S_2}{24}$$
$$18\ S_2=24\times 33$$
$$S_2=44 \text{ r.p.m.}, \textit{ Ans.}$$

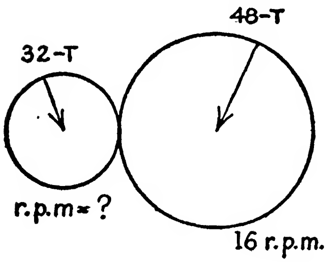
*Exercise 24.*

1. The resistance ($R$) in an electric circuit varies inversely as the current ($C$). If $R_1=40$ and $C_1=54$, find $C_2$ when $R_2=180$; find $R_2$ when $C_2=36$.
2. The volume of a gas at constant temperature varies inversely as its pressure. If the volume of the gas in a certain cylinder is 720 cu. in. at a pressure of 20 lb. per sq. in., what will be its volume under a pressure of 25 lb. per sq. in.? What pressure will be required to reduce its volume to 200 cu. in.?

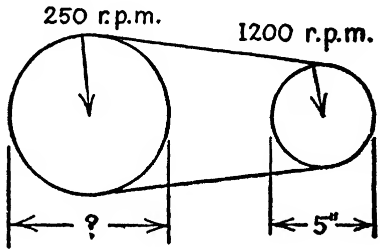
3. The so-called principle of moments, or balanced turning tendencies, follows the law of inverse proportion; for $\frac{w_1}{w_2}=\frac{d_2}{d_1}$, or $w_1d_1=w_2d_2$. Find $w_2$ if $w_1=4$ lb., $d_1=6$ in., and $d_2=8$ in.; find $d_2$ if $w_1=24$ lb., $w_2=30$ lb., and $d_1=10$ in.




4. The density of a substance varies inversely as its volume when the mass (weight) is constant, or $D_1 : D_2=V_2 : V_1$. If the density of a gas is .0015 when its volume is 8000 cu. cm., what will its density be when it has been compressed to a volume of 5000 cu. cm.?

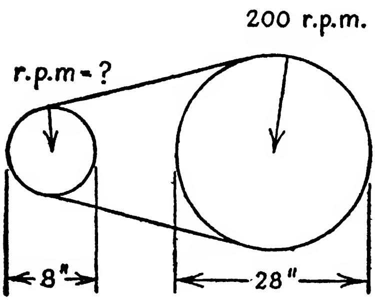



Ex. 5

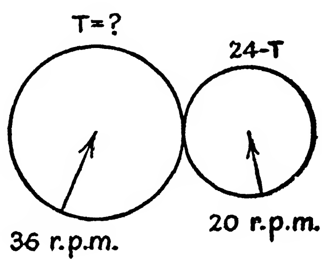



Ex. 6

5. Find the number of r.p.m. of the smaller gear.
6. Find the diameter of the larger pulley.




Ex. 7




Ex. 8

7. Find the speed of the smaller pulley.
8. Find, to the nearest whole number, the number of teeth in the larger gear.
9. If a driving pulley has a diameter of 20″ and its speed is 750 r.p.m., what is the speed of a 6″ driven pulley?

10. A circular buffer 6″ in diameter should revolve at 1800 r.p.m. If it is driven by a line shaft revolving at 400 r.p.m., find the diameter of the driving pulley on the line shaft that will be needed to obtain the desired speed of the buffer.

# CHAPTER II

# ELEMENTS OF ALGEBRA

*William L. Schaaf*

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## 7. NUMBERS AND SYMBOLS

**The Language of Algebra.** The methods of algebra are essentially an extension of arithmetic. In other words, the numbers, symbols and operations used in algebra are the same as those used in arithmetic, only they are *more general* in character. This means (1) that *letters* as well as numbers are used to represent quantities, and (2) that numbers are regarded as having *quality* as well as quantity.

Consider the use of letters as symbols of quantity. In arithmetic, we say a force of 10 lb.; in algebra, we say a force of $F$ lb., and we think of the letter $F$ as having *various numerical values,* either one after another, or even "all at once." Or again, in arithmetic we say a bolt has a diameter of 1½″; in the language of algebra we say it has a diameter of $d$ inches; etc.

The following will illustrate how *verbal statements* are translated into *algebraic symbols:*

| *Verbal statement* | *Algebraic formulation* |
|---|---|
| (1) Four times a number. | $4n$ |
| (2) Sum of two numbers decreased by a third number. | $a+b-c$ |
| (3) Three times a number increased by twice a second number. | $3k+2m$ |
| (4) Sum of two numbers divided by 5 times a third number. | $\frac{x+y}{5z}$ |
| (5) Twice the product of two numbers increased by ⅔ of a third number. | $2ab+\frac{2}{3}c$ |
| (6) Ten times a number decreased by six. | $10n-6$ |

### *Exercise 25.*

"Translate" into words each of the following algebraic expressions:

1. $2a+3b$
2. $5x-\frac{1}{2}y$
3. $\dfrac{a-b+c}{3}$
4. $\frac{2}{3}k+8$
5. $P-\dfrac{1}{N}$
6. $5Rh$
7. $\dfrac{m-n}{2p}$
8. $\frac{9}{5}C+32$
9. $2lw+2lh+2wh$
10. $\dfrac{H-h}{T-t}$

**Substitution.** The process of finding the numerical value of an algebraic expression for certain specific values of the letters that occur in the expression is known as *substitution.*

EXAMPLE 1: Find the value of $\dfrac{\pi D}{4}h$, when $h=12$, $D=3.5$, and $\pi=3\frac{1}{7}$.

SOLUTION: $\frac{1}{4} \cdot \frac{22}{7} \cdot \frac{7}{2} \cdot 12=33$, *Ans.*

EXAMPLE 2: Find the value of $2ak-\frac{3}{4}m$, when $a=20$, $k=2.5$, and $m=1.6$.

SOLUTION: $(2)(20)(\frac{5}{2})-(\frac{3}{4})(1.6)=100-1.2=98.8$, *Ans.*

### *Exercise 26.*

Find the numerical value of each of the following:

1. $2x+3y$, when $x=4\frac{1}{2}$ and $y=0.8$.
2. $\frac{3}{4}h-2$, when $h=16.44$.
3. $\dfrac{p+q}{3r}$, when $p=10.3$, $q=8.6$, and $r=.07$.
4. $2\frac{1}{4}D+1\frac{3}{4}$, when $D=.816$.
5. $\frac{4}{3}\pi R^3$, when $\pi=3\frac{1}{7}$ and $R=3.5$.
6. $0.3707p-0.0052$, when $p=\frac{1}{6}$.
7. $\frac{9}{5}C+32$, when $C=99.5$.

Complete each of the following tables of values as indicated:

(8)

| $D$ | $D/2$ |
|---|---|
| 4 | ? |
| 6 | ? |
| 8 | ? |
| 10 | ? |
| 15 | ? |

(9)

| $k$ | $8k$ |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |

(10)

| $H$ | $.75H$ |
|---|---|
| 4 | |
| 8 | |
| 10 | |
| 12 | |
| 24 | |

(11)

| $N$ | $\frac{1}{8}N$ |
|---|---|
| 0 | |
| 2 | |
| 4 | |
| 6 | |
| 10 | |

(12)

| $R$ | $\frac{4}{3}R$ |
|---|---|
| 2 | |
| 4 | |
| 6 | |
| 8 | |
| 12 | |

(13)

| $p$ | $\frac{2}{3}p$ |
|---|---|
| 1 | |
| 3 | |
| 5 | |
| 7 | |
| 10 | |

(14)

| $h$ | $\frac{h}{3.14}$ |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 10 | |

(15)

| $x$ | $4\frac{1}{2}x$ |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 4 | |
| 10 | |

**Positive and Negative Quantities.** The second feature which distinguishes algebra from arithmetic is the use of *negative* numbers as well as positive numbers. Accordingly, all numbers and letters are assumed to be either positive or negative (except zero), and are designated, respectively, as + or —; thus +5, —2, +½, —0.15, etc. The essential significance of these signs is to denote *oppositeness,* i.e., oppositeness of direction on a number scale, as suggested below:

-5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5

*Addition* of signed numbers in algebra is equivalent to combining "steps" or "intervals" along a number scale, the signs of the numbers indicating the "direction" from the zero point taken in each step. The "sum" of an algebraic addition is thus the *net result* of combining two or more such steps. For example, by algebraic addition, we get:

$$\begin{array}{ccccccc}
+3 & +4 & -5 & -3 & +2 & -1 & +4 \\
\underline{+2} & \underline{-3} & \underline{+2} & \underline{-2} & \underline{-3} & \underline{+3} & \underline{-4} \\
+5 & +1 & -3 & -5 & -1 & +2 & 0
\end{array}$$

If no sign appears before a quantity, it is assumed to be positive.

RULE I: *To add two signed numbers whose signs are* alike, *find their arithmetic sum and prefix the same sign that both have.*

RULE II: *To add two signed numbers whose signs are* opposite, *find their arithmetic difference, and then prefix the sign of the (numerically) larger quantity.*

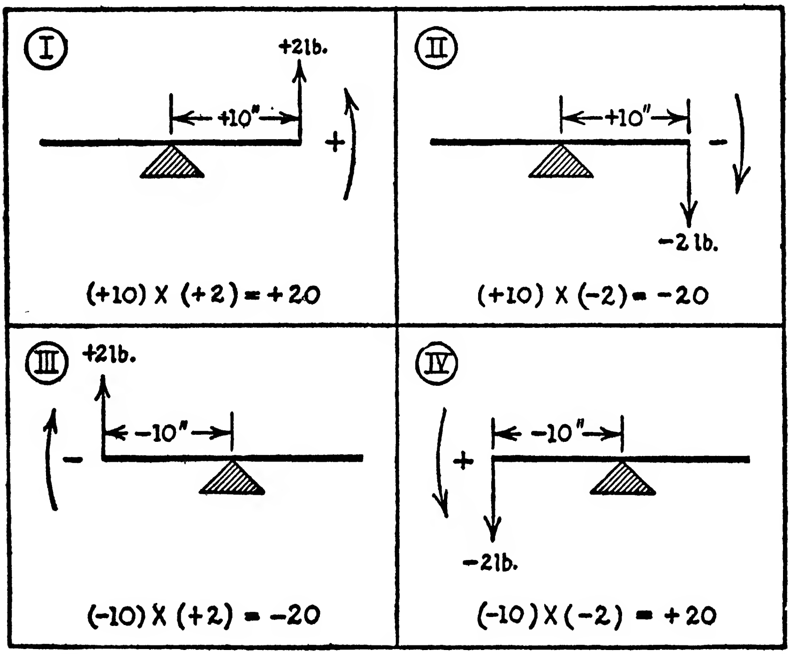
Subtraction being the opposite or inverse process of addition, it might be expected that we also reverse something when we subtract in algebra; we do. To subtract a quantity in algebra, we add the same quantity *with the opposite sign.* Hence

RULE III: *To subtract one signed number from another, change the sign of the subtrahend; then add them algebraically.*

For example, when subtracting the lower quantity from the upper in each case, we get:

| | | | | | | |
|---|---|---|---|---|---|---|
| +6 | +4 | −7 | −8 | +3 | −5 | −6 |
| +4 | −3 | +2 | −8 | −7 | +9 | −3 |
| +2 | +7 | −9 | 0 | +10 | −14 | −3 |

**Multiplication and Division with Signed Numbers.** The rules concerning signs when multiplying and dividing signed numbers are simple, and, while they may seem a bit arbitrary, their reasonableness may be seen by studying the accompanying diagrams of a lever (or see-saw), where it




is agreed that distance to the right of P are +, and to the left, —; where it is also agreed that downward pulls are negative (—) and upward pushes positive (+); and finally, that clockwise turning is negative and counterclockwise, positive.

RULE IV: *In multiplication, like signs give a positive product, and unlike signs give a negative product.*

For division, the same rule holds true: if the signs of the dividend and divisor (or numerator and denominator) are alike, the quotient is positive; if unlike, the quotient is negative.

*Exercise 27.*

1. Add:

| | | | | | | |
|---|---|---|---|---|---|---|
| $+14$ | $+22$ | $+17$ | $-31$ | $-12$ | $+15$ | $-18$ |
| $-\ 8$ | $+16$ | $-17$ | $+12$ | $-10$ | $-27$ | $30$ |

2. Subtract:

| | | | | | | |
|---|---|---|---|---|---|---|
| $+15$ | $+18$ | $-16$ | $+32$ | $-14$ | $-18$ | $-25$ |
| $-\ 8$ | $+\ 5$ | $-10$ | $+32$ | $-12$ | $18$ | $-25$ |

3. Multiply:

| | | |
|---|---|---|
| $(+3)(-21)$ | $(+16)(+\frac{1}{8})$ | $(-10m)\times(-3)$ |
| $(-6)(-10)$ | $(-\frac{1}{2})(-12.4)$ | $(4)\times(-3a)$ |
| $(-15)(+4)$ | $(-6)\times(12k)$ | $(-8)\times(6r)$ |

4. Divide:

| | | |
|---|---|---|
| $(+18)\div(-2)$ | $(+24)\div(+2)$ | $(+12)\div(-6)$ |
| $(-27)\div(+3)$ | $(+50)\div(-25)$ | $24\div(-3a)$ |
| $(-15)\div(-5)$ | $(-16)\div(-16)$ | $(-48x)\div 6$ |

**Addition and Subtraction of Similar Terms.** In algebra, quantities to be added or subtracted are known as *terms. Similar terms* are terms similarly composed of the same letters; thus, for example, $3a$ and $5a$ are similar terms; so are $3xy$, $-2xy$ and $6xy$. *Only similar terms can be added and subtracted.* The sum of $4n$, $-3n$, and $5n$ is $6n$; but $4n$ and $3p$ cannot actually be added (until numerical values are substituted for both $n$ and $p$). In terms like these the number before a letter (or group of letters mutiplied together) is known as the numerical coefficient of the term.

RULE V: *To add (or subtract) similar terms, add (or subtract) their numerical coefficients, and annex the same letters to the new coefficient.*

*Exercise 28.*

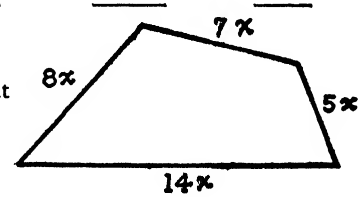
1. Add:

| | | | | |
|---|---|---|---|---|
| $15p$ | $+28r$ | $-\ 3m$ | $+16.4h$ | $-\ 8\frac{1}{2}x$ |
| $6p$ | $-\ 2r$ | $12m$ | $-\ 7.8h$ | $-14x$ |

2. Subtract:

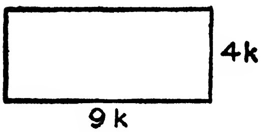
| | | | | |
|---|---|---|---|---|
| $12k$ | $-20a$ | $+15xy$ | $-18lw$ | $+\frac{1}{2}ab$ |
| $8k$ | $-14a$ | $-11xy$ | $+18lw$ | $-2ab$ |

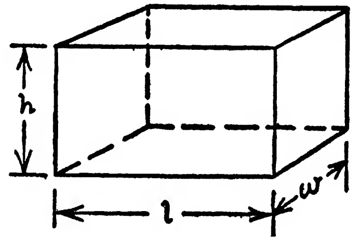
3. Find the perimeter of the figure at the right.

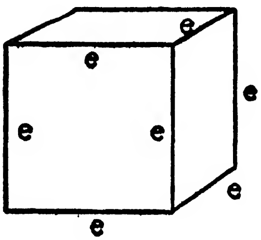


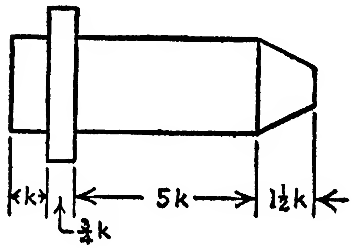
4. What is the perimeter of the rectangle shown below.

5. Find the sum of the twelve edges of the rectangular solid shown below.



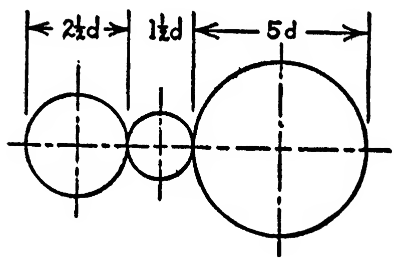




6. What is the sum of all the edges of the cube?

7. What is the overall length of the piece of work shown?

8. Find the overall diameter of this train of gears.




**Multiplying and Dividing Literal Numbers.** Although only similar terms can be added and subtracted, the operations of multiplication and division can be performed on any number of terms, whether similar or not. Thus, $2a \times 3b = 6ab$; $5mh \times 3r = 15mhr$. Or again, $12a \div 4a = 3$; $16xy \div 8x = 2y$; etc. Furthermore, to show how similar terms are multiplied and divided, study the following examples:

$$a \times a = a^2$$
$$b \times b \times b = b^3$$
$$3a \times 5a = 15a^2$$
$$4p \times 3pq = 12p^2q$$
$$5h^2r \times 2hr^3 = 10h^3r^4$$

$$\frac{6a^2}{2a} = 3a$$
$$\frac{-12m^2x}{2mx^3} = \frac{-6m}{x^2}$$
$$\frac{-8a^2b^3}{-4ab^2} = 2ab$$

NOTE: In algebra, the multiplication sign ($\times$) is frequently omitted between two letters to be multiplied together; instead, a dot is placed between them, or nothing at all, to indicate multiplication. Thus

$$(2m)\times(r)=2m\cdot r=2mr.$$

*Exercise 29.*

1. Find the product of:

| | | |
|---|---|---|
| $8\times 4m$ | $3\frac{1}{7}\times 21h$ | $4a^2b\cdot 3ab^2$ |
| $10\times 3k$ | $\frac{2}{3}s\cdot\frac{1}{2}t$ | $5x^3\cdot x^4$ |
| $5p\times 3q$ | $\frac{1}{3}r^2\cdot 15h$ | $m^2\cdot m^3$ |
| $6h\times 2ab$ | $4\,R^2\cdot\frac{1}{3}R$ | $12xy\cdot\frac{1}{2}xy$ |
| $4a^2\times 3ab$ | $3.8at\times 2at^2$ | $(\frac{1}{2}a^3)\cdot(\frac{1}{3}a^2)$ |

2. Divide:

| | | |
|---|---|---|
| $14m\div 2$ | $240k\div 32$ | $15r^2\div 5rs^3$ |
| $32x\div 4x$ | $35y^2\div 5y$ | $-8cd^3\div c^2d$ |
| $24p\div p$ | $12a^3\div ab$ | $2\pi R^2h\div\pi R$ |
| $15\div 5x$ | $20xy\div 4xy$ | $-10a^3b^6\div 5ab^4$ |

**Terms and Factors.** The distinction about to be made between *terms* and *factors* is very important. Thus in the expression $(2)(a+b)$, one factor is "2" and the other is the *entire* quantity "$a+b$." *Factors* are quantities which are to be multiplied or divided; *terms* are quantities that are to be added or subtracted. Thus, in the expression $2a+2b$ there are *two terms,* viz., "$2a$" and "$2b$"; each of these terms consists of two factors, to wit, 2 and $a$, and 2 and $b$, respectively. In the first instance the parenthesis ( ) indicates that the quantity enclosed within it is to be regarded as *all one quantity* so far as the operations of multiplication (or division) are applied to it. In short, whatever the "$a$" is multiplied by, the "$b$" must also be multiplied by. Or

$$3(a+b)=3a+3b;$$
$$\tfrac{1}{2}(p+q-m)=\tfrac{1}{2}p+\tfrac{1}{2}q-\tfrac{1}{2}m;\ \text{etc.}$$

*Exercise 30.*

Write each of the following without the parentheses:

1. $2(l+w)$
2. $k(a+b-c)$
3. $a+(n-1)d$
4. $I(R+r)$
5. $P(1+RT)$
6. $S=\frac{n}{2}(a+l)$
7. $\frac{h}{2}(B+b)$
8. $\pi(R^2-r^2)$
9. $L(t_0-t_1)$
10. $2\pi r(l+r)$
11. $\frac{n}{2}[2a+(n-1)d]$
12. $V=\frac{h}{6}(b_1+b_2+4M)$

Factor each of the following expressions:

| | |
|---|---|
| 13. $p+pi$ | 17. $P+Prt$ |
| 14. $hp-hq$ | 18. $\frac{1}{2}B_1h+\frac{1}{2}B_2h$ |
| 15. $I^2R+I^2r$ | 19. $\pi Dl+\pi D$ |
| 16. $2ab-2ac$ | 20. $2\pi Rl+2\pi R^2$ |

Find the numerical value of each of the following:

21. $\frac{5}{9}(F-32)$, when $F=212$; when $F=68$.

22. $\frac{1}{2}h(B_1+B_2)$, when $h=6\frac{1}{2}$, $B_1=10.4$, and $B_2=5.6$.

23. $\dfrac{12(D-d)}{L}$, when $D=8.5$, $d=6.25$, and $L=48$.

24. $2\pi h(R^2-r^2)$, when $R=10$, $r=6$, $h=21$, and $\pi=3\frac{1}{7}$.

**Squaring a Number.** In order to find the area of a square we simply multiply the length of the side *by itself.* This is called squaring a number. The square, or second power, of a number is the product of the number multiplied by itself (not multiplied by 2); thus $3^2=3\times3=9$, or $10^2=10\times10=100$. If an indicated product contains two equal factors, it is the "square" of either one of them.

**The Cube of a Number.** Similarly, if a number is multiplied *by itself 3 times* (not multiplied by 3), the product is called the *cube* of the number; this is suggested by the fact that the volume of a cube equals the edge multiplied by itself three times. An indicated product having three equal factors is called the "cube" of any one of them. Thus, $2^3=2\times2\times2=8$; $5^3=5\times5\times5=125$; $10^3=10\times10\times10=1000$; etc.

*Exercise 31.*

Find the value of each of the following:

| | | | |
|---|---|---|---|
| 1. $8^2=?$ | 5. $1^3=?$ | 9. $(\frac{1}{2})^2=?$ | 13. $(\frac{3}{4})^3=?$ |
| 2. $16^2=?$ | 6. $3^3=?$ | 10. $(\frac{2}{3})^2=?$ | 14. $(.01)^2=?$ |
| 3. $6^3=?$ | 7. $20^2=?$ | 11. $(\frac{1}{4})^3=?$ | 15. $(.01)^3=?$ |
| 4. $12^3=?$ | 8. $7^3=?$ | 12. $(\frac{1}{10})^2=?$ | 16. $(.001)^2=?$ |

**Other Powers.** A number may be "raised to any power" desired. Thus:

$3^4=3\times3\times3\times3=81$ ("fourth power of 3")
$2^5=2\times2\times2\times2\times2=32$ ("fifth power of 2")
$10^6=10\times10\times10\times10\times10\times10=1{,}000{,}000$ ("sixth power of 10")

As a matter of fact, using various "powers" of 10 is a convenient device employed by scientists and engineers. Let us first study the following table of powers of the base 10:

| | |
|---|---|
| $10^1=10$ | $10^{-1}=0.1$ |
| $10^2=100$ | $10^{-2}=0.01$ |
| $10^3=1000$ | $10^{-3}=0.001$ |
| $10^4=10{,}000$ | $10^{-4}=0.0001$ |
| $10^5=100{,}000$ | $10^{-5}=0.00001$ |
| $10^6=1{,}000{,}000$, etc. | $10^{-6}=0.000001$, etc. |

Using these values, we can "abbreviate" a number like 290,000 as $29\times10^4$, since $29\times10^4=29\times10{,}000=290{,}000$. Similarly, a large number like 3,920,000,000 may be expressed more conveniently as $3.92\times10^9$, since $10^9=1{,}000{,}000{,}000$. Very small numbers, like .00000057 may be written as $57\times10^{-8}$; and so on.

*Exercise 32.*

Express each of the following in full:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | $34\times10^5$ | 4. | $43\times10^{10}$ | 7. | $8.62\times10^8$ | 10. | $23\times10^{-8}$ |
| 2. | $62\times10^8$ | 5. | $5.2\times10^6$ | 8. | $24.63\times10^{12}$ | 11. | $4.9\times10^{-9}$ |
| 3. | $91\times10^9$ | 6. | $13.8\times10^{11}$ | 9. | $35\times10^{-6}$ | 12. | $32.6\times10^{-11}$ |

Express each of the following as a power of 10:

| | | | |
|---|---|---|---|
| 13. 37,000,000 | 15. 12,400,000 | 17. 0.00004 | 19. 0.000392 |
| 14. 5,800,000 | 16. 4,900,000,000 | 18. 0.00000028 | 20. 0.00000000076 |

21. Astronomers use a unit known as the "light year," which is the distance traveled by light in one year. If this distance is equal to $5.8825\times10^{12}$ miles, express this distance without using a power of 10.
22. Scientists have measured the wave length of sodium light and found it to be 0.0005893 millimeters. Express this in abbreviated standard form.

**Laws of Exponents.** When a quantity, whether a number or a letter, is raised to a power, the small number which indicates the power to which it is to be raised, and which is written to the upper right of it, is called an *exponent.* Thus in $25^2$, the exponent is *2;* in $3x^5$, the exponent is *5.* Letters, too, may be used as exponents; thus, in $10^m$, $4a^x$, and $P^{k+1}$, the exponents are *m, x,* and $(k+1)$, *respectively.* When powers are to be combined by multiplication or division, the following principles must be observed:

| *Principle* | *Illustration* |
|---|---|
| (1) $a^m\cdot a^n=a^{m+n}$ | (1) $x^4\cdot x^3=x^{4+3}=x^7$ |
| (2) $a^m\div a^n=a^{m-n}$ | (2) $y^5\div y^2=y^{5-2}=y^3$ |
| (3) $(a^m)^n=a^{mn}$ | (3) $(p^3)^2=p^{(3)\,(2)}=p^6$ |
| (4) $(ab)^m=a^mb^n$ | (4) $(ar)^4=a^4r^4$ |

**Meaning of a Root.** The operation which is the inverse (opposite) of raising to a power is called "taking the root." In other words, finding the square root of a quantity means finding the two equal factors which multiplied give that quantity; to find a cube root means finding the three equal factors of that product; etc.

EXAMPLES

| *Raising to a Power* | *Expressing the Root* |
|---|---|
| $a^2=a\times a=P$ | $\sqrt{P}=a$ |
| $x^3=x\,.\,x\,.\,x=Q$ | $\sqrt[3]{Q}=x$ |
| $k^5=k\,.\,k\,.\,k\,.\,k\,.\,k=R$ | $\sqrt[5]{R}=k$ |

Roots may also be expressed by using *fractional exponents,* as shown:

$$\sqrt{m}=m^{1/2} \qquad \sqrt[4]{x}=x^{1/4}$$
$$\sqrt[3]{a}=a^{1/3} \qquad \sqrt[5]{p}=p^{1/5}$$
$$\sqrt[3]{y^2}=y^{2/3}$$
$$\sqrt{A^3}=A^{3/2}$$

**Negative Exponents.** We have already seen that $10^{-1}=1/10$ or 0.1; $10^{-2}=1/100$, or 0.01; etc. A quantity with a *negative exponent* indicates that *that quantity with the same positive exponent is to be divided into one,* i.e., it indicates the *reciprocal* of that power; thus

$$a^{-1}=\frac{1}{a};\quad x^{-2}=\frac{1}{x^2};\quad p^{-5/3}=\frac{1}{\sqrt[3]{p^5}}$$

This may be better understood by studying the following summary:

$$a^5=aaaaa \qquad a^{-1}=\frac{1}{a}$$
$$a^4=aaaa \qquad a^{-2}=\frac{1}{aa}=\frac{1}{a^2}$$
$$a^3=aaa \qquad a^{-3}=\frac{1}{aaa}=\frac{1}{a^3}$$
$$a^2=aa \qquad a^{-4}=\frac{1}{aaaa}=\frac{1}{a^4}$$
$$a^1=a \qquad a^{-5}=\frac{1}{aaaaa}=\frac{1}{a^5}$$

The reason why a negative exponent causes a factor to appear in the denominator may be seen by applying principle No. 2 above. Thus

$$\frac{a^3}{a^5}=a^{3-5}=a^{-2};$$

$$\text{but } \frac{a^3}{a^5}=\frac{\overset{1\,1\,1}{\cancel{a}\cancel{a}\cancel{a}}}{\cancel{a}\cancel{a}\cancel{a}aa}=\frac{1}{a^2};$$

$$\text{therefore } a^{-2}=\frac{1}{a^2}.$$

$$\text{Similarly, } \frac{x}{x^4}=x^{1-4}=x^{-3};$$

$$\text{but } \frac{x}{x^4}=\frac{\overset{1}{\cancel{x}}}{\cancel{x}xxx}=\frac{1}{x^3};$$

$$\text{therefore } x^{-3}=\frac{1}{x^3}.$$

The same principle also shows why any quantity with a *zero exponent* must always be equal to 1, no matter what the quantity itself may be; thus

$$\frac{a^5}{a^5}=a^{5-5}=a^0; \text{ but } \frac{a^5}{a^5}=1; \text{ hence } a^0=1$$

$$\frac{x^{13}}{x^{13}}=x^{13-13}=x^0; \text{ but } \frac{x^{13}}{x^{13}}=1; \text{ hence } x^0=1.$$

**Roots and Powers of Fractions.** Just to make sure, the reader is reminded that in raising a fraction to a power, both numerator and denominator are raised to that power; thus

$$(2/3)^2=\frac{2^2}{3^2}=4/9;\ (3/10)^3=.027;\ \left(\frac{2\pi R^2}{h}\right)^2=\frac{4\pi^2R^4}{h^2}.$$

Likewise when taking a root:

$$\sqrt{\frac{9}{16}}=\frac{\sqrt{9}}{\sqrt{16}}=\frac{3}{4};\ \sqrt[3]{\frac{1}{8}}=\frac{1}{2};\ \sqrt{\frac{2E}{m^2}}=\frac{\sqrt{2E}}{m}$$

*Exercise 33.*

1. Write the following without using exponents:
 $a^{1/2}$; $x^{1/4}$; $(5a)^{1/2}$; $p^{2/3}$; $A^{3/5}$; $(mn)^{1/3}$.

2. Write the following without using radicals:

$$\sqrt{p};\quad \sqrt[4]{k};\quad \sqrt[3]{a^2};\quad \sqrt[5]{10x};\quad \sqrt{\frac{2s}{g}};\quad \sqrt{\frac{A}{\pi}}.$$

3. Find the value of the following:

$x^{1/2}$, when $x=144$.
$k^{1/3}$, when $k=27/8$.
$a^{2/3}$, when $a=8$.
$x^{2/3}$, when $x=8$.
$p^{1/2}$, when $p=36a^2$.
$3a^2b^3$, when $a=2$ and $b=1$.
$2x^3y^2$, when $x=3$ and $y=2$.
$(2x^3)^2$, when $x=10$.

4. What is the numerical value of each of the following?

| | | | |
|---|---|---|---|
| $25^{-1/2}$ | $(.04)^{1/2}$ | $(.027)^{1/3}$ | $10^{-3}$ |
| $4^{3/2}$ | $64^{2/3}$ | $16^{3/4}$ | $(5x)^{-2}$ |
| $27^{2/3}$ | $32^{1/5}$ | $8^{-1/3}$ | $(16a^4)^{-1/2}$ |

## 8. FORMULAS AND EQUATIONS

**Meaning of a Formula.** Many of the computations used in shop problems are either simplified or more clearly understood by the use of formulas. A *formula* is simply a mathematical statement of a principle or a rule describing the relation between two or more quantities. This mathematical statement shows that there is an *equality* between certain quantities; in other words, the formula translates a verbal rule into algebraic symbols. Thus a formula is very similar to an *equation.* For example, since there are 12 inches to every foot, this verbal rule can be stated as a formula by writing $I=12F$; or, since the percentage equals the base multiplied by the rate, we have the formula $P=BR$; or again, if in the lever, one weight multiplied by its distance from the support *balances* (i.e., equals) the other weight multiplied by *its* distance, then we can write this as a formula by saying: $w_1d_1=w_2d_2$.

**Evaluating Formulas.** It will be seen that a formula may involve two, three, four or even more quantities. If the formula expressing the relation or connection between these quantities is known, and if a particular value is known for every quantity in the formula *except one,* then the value of that remaining one is easily found. This is sometimes called "evaluating a formula," or "substituting in a formula."

EXAMPLE 1: If $P=2(l+w)$, find $P$ when $l=7.9$ and $w=5.2$.

SOLUTION: $P=2(7.9+5.2)$
$=2(13.1)=26.2$, *Ans.*

EXAMPLE 2: If $A=\frac{1}{2}h(B_1+B_2)$, what is the value of $A$ when $h=8$, $B_1=4\frac{1}{2}$, and $B_2=6\frac{1}{2}$?

SOLUTION: $A=\frac{1}{2}(8)(4\frac{1}{2}+6\frac{1}{2})$
$=\frac{1}{2}(8)(11)=44$, *Ans.*

EXAMPLE 3: If $S=\frac{1}{2}\,at^2$, find $S$ when $a=32.2$ and $t=3$; also when $a=980$ and $t=10$.

SOLUTION: $S=\frac{1}{2}(32.2)\ (3)^2$
$=\frac{1}{2}(32.2)\ (9)=144.9$, *Ans.*
$S=\frac{1}{2}(980)\ (10)^2=49{,}000$, *Ans.*

EXAMPLE 4: If $C=\frac{5}{9}(F-32)$, find $C$ when $F=212$; when $F=32$.

SOLUTION: $C=\frac{5}{9}(212-32)=\frac{5}{9}\ (180)=100$, *Ans.*
$C=\frac{5}{9}(32-32)=\frac{5}{9}\ (0)=0$, *Ans.*

*Exercise 34.*

1. If $V=lwh$, find $V$ when $l=14''$, $w=6\frac{1}{2}''$ and $h=4''$.
2. If $A=6e^2$, find $A$ when $e=2\frac{1}{2}''$.
3. In the formula $V=v_0+gt$, find the value of $V$ when $v_0=200$, $g=32$, and $t=8$.
4. Given the relation $F=\frac{9}{5}C+32$, what is the value of $F$ when $C=18^0$?
5. A formula for the amount of money due on a loan at simple interest is: $A=P\left(1+\frac{nr}{12}\right)$. Find $A$ when $P=\$500$, $r=.04$, and $n=6$ months.
6. An electric current (I) is expressed by the formula $I=\frac{E}{R+r}$; find $I$ when $R=85$, $r=3$, and $E=220$.
7. Under certain conditions the energy of a moving body is given by $E=\frac{Mv^2}{2g}$; find $E$ when $M=2000$, $v=80$, and $g=32$.
8. If $I=\frac{\pi abC^2}{6}$, what is the value of $I$ when $a=3.6$, $b=2.1$, $C=2$, and $\pi=3\frac{1}{7}$?
9. In designing modern automobile highways, the following formula relating to motor trucks is sometimes used: $W=k(L+40)$, where $W$ =total gross weight of truck with its load, $L$=distance in feet between first and last axles of the truck, or truck and trailer. If the value of $k$ in a certain state=720, and $L$=42 ft., find the value of $W$.
10. The heat generated in an electric circuit is expressed by the formula $H=0.24Ri^2t$. If $R$, the resistance, equals 55 ohms; $i$, the current, equals 2 amperes; and $t$, the time, equals $\frac{1}{2}$ hour, find the amount of heat ($H$) produced (expressed in calories).

11. A formula sometimes used for finding the horse power rating of a gas engine is: H.P.$=\dfrac{D^2n}{2.5}$, where $D$=diameter of the cylinder in inches and $n$=number of cylinders, provided the piston speed is 1000 ft. per min. Find the H.P. of an engine having 12 cylinders with a 4″-bore, running at 2000 ft. per min.
12. Marine engineers find the "wetted surface" of a ship by using the formula $A=15\sqrt{Dl}$, where $D$=displacement in tons and $l$=length of the ship in feet. If a ship 900 ft. long displaces 81,000 tons, find the wetted surface $A$ (in sq. ft.).

**Simple Equations.** As already stated, an equation is very similar to a formula. The chief difference, as far as we are concerned here, is this: in a formula there are at least two "quantities," or literal numbers, whereas in a simple equation there is only one such literal number. Thus:

| *Formulas* | *Equations* |
|---|---|
| $A=lw$ | $12x=5-3x$ |
| $V=\frac{1}{3}Bh$ | $\frac{3}{5}x=42$ |
| $\dfrac{1}{p}+\dfrac{1}{q}=\dfrac{2}{R}$ | $\dfrac{x}{3}+\dfrac{x}{2}=12\frac{1}{2}$ |

It turns out, therefore, for reasons that will become clearer as we go along, that in a simple equation there is *only one* numerical value which will hold true for the literal number; in the case of a formula, however, there are indefinitely many combinations of values for the literal quantities which will make the formula hold true. The process of finding the particular value which holds true for the literal quantity in any equation is called "solving the equation"; that numerical value of the letter is called the "solution," and it is said to "satisfy" the equation. For example, if it is known that $6x-17=4x+13$, it is possible to find a value which *satisfies* this equation; the desired value is 15. For if $x=15$, then substituting 15 for $x$ in the equation we obtain:

$$(6)(15)-17=(4)(15)+13$$
$$90-17=60+13$$
$$73\equiv73$$

This shows that the equality expressed by the original equation really holds true when $x$ has the numerical value 15, since 73 is identical with 73, or both sides of the equation have been shown to be identical.

**Solving an Equation by Division.** Certain types of equations are readily solved by dividing both sides of the equation by an appropriate number.

EXAMPLE 1: Solve the equation $6x=51$.

SOLUTION: Dividing each "side" of the equation by 6 we obtain

$$x=\frac{51}{6}$$

or $x=8\frac{1}{2}$, *Ans.*

EXAMPLE 2: Solve for $y$: $3\frac{1}{2}y=28$

SOLUTION: Dividing both sides by $3\frac{1}{2}$, we obtain

$$y=\frac{28}{3\frac{1}{2}}$$

or $y=(28)(\frac{2}{7})$

$y=8$, *Ans.*

**Solving an Equation by Multiplication.** Sometimes an equation may be conveniently solved by multiplication instead; each side of an equation may be multiplied by any number, provided it is the *same* number.

EXAMPLE 1: Solve: $\frac{x}{14}=3$

SOLUTION: Multiplying both sides by 14 we obtain

$x=(14)(3)$

or $x=42$, *Ans.*

EXAMPLE 2: Solve for $n$ the equation $\frac{n}{.6}=25$

SOLUTION: Multiplying both sides by .6 we obtain

$n=(25)(.6)$

or $n=15$, *Ans.*

*Exercise 35.*

Solve each of the following equations for the "unknown" letter:

1. $5n=65$
2. $\frac{x}{4}=8$
3. $100x=35$
4. $\frac{1}{8}p=2.5$
5. $84=7k$
6. $\frac{y}{9}=1\frac{2}{3}$
7. $3\frac{1}{2}x=49$
8. $.05k=16$
9. $\frac{n}{.4}=30$
10. $\frac{2x}{3}=24$
11. $.3x=4.8$
12. $\frac{3}{4}h=120$

**Changing the Subject of a Formula by Multiplication or Division.** The same procedure as used for equations can also be applied to solving a formula for any particular letter desired. The other letters are simply regarded as numbers. This is called "changing the subject of the formula," or solving for a particular letter "in terms of the other letters."

EXAMPLE 1: Solve the formula $F=ma$ for $m$.

SOLUTION: Dividing both sides by $a$:

$$\frac{F}{a}=m, \text{ or } m=\frac{F}{a}, \textit{ Ans.}$$

EXAMPLE 2: In the formula $\frac{E}{I}=nR$, express $I$ in terms of $E$, $n$ and $R$.

SOLUTION: Multiplying both sides by $I$:

$$E=InR$$

Dividing both sides by $nR$:

$$\frac{E}{nR}=I, \text{ or } I=\frac{E}{nR}, \textit{ Ans.}$$

*Exercise 36.*

Solve each of the following formulas for the quantity specified:

1. $A=lw$; solve for $l$.
2. $D=\frac{M}{V}$; solve for $M$.
3. $D=RT$; solve for $T$.
4. $I=\frac{E}{R}$; solve for $R$.
5. $C=2\pi R$; solve for $R$.
6. $d=1.4s$; solve for $s$.
7. $PV=kT$; solve for $P$.
8. $I=PRT$; solve for $T$.
9. $A=\frac{1}{2}bh$; solve for $h$.
10. $S=2\pi rh$; solve for $r$.
11. $l_1 w_1=l_2 w_2$; solve for $w_2$
12. $S=\frac{1}{2}gt^2$; solve for $t^2$.

**Solving an Equation by Addition or Subtraction.** Frequently an equation is of such a form that its solution may be effected by adding or subtracting appropriate numbers, as shown below. Any number may be added to, or subtracted from, both sides of an equation; but it must be the same number.

EXAMPLE 1: Solve for $x$: $x+10=17$

SOLUTION: Subtracting:

$$\begin{array}{rcl} x+10 &=& 17 \\ 10 &=& 10 \\ \hline x &=& 7, \textit{ Ans.} \end{array}$$

EXAMPLE 2: Solve for $k$: $k-6\frac{1}{2}=12$

SOLUTION: Adding:

$$\begin{array}{rcl} k-6\frac{1}{2} &=& 12 \\ 6\frac{1}{2} &=& 6\frac{1}{2} \\ \hline k &=& 18\frac{1}{2}, \textit{ Ans.} \end{array}$$

### *Exercise 37.*

Solve each of the following equations for the unknown letter:

| | |
|---|---|
| 1. $n+12=34$ | 7. $14=x+10\frac{1}{2}$ |
| 2. $30=16+k$ | 8. $n-3.5=7.2$ |
| 3. $x-25=13$ | 9. $y+6.4=8\frac{1}{2}$ |
| 4. $60=y-35$ | 10. $x-3\frac{1}{2}=5\frac{3}{4}$ |
| 5. $28+p=42$ | 11. $18=x-4\frac{1}{2}$ |
| 6. $h-16=20$ | 12. $40.2=78.5-x$ |

**Solving Any Kind of a Simple Equation.** By "any kind" of a simple equation we simply mean an equation in which several operations may be necessary to find the solution. The procedure is illustrated by the following:

EXAMPLE: Solve: $5x+8=2x+20$

SOLUTION:
$$5x+8-2x=20$$
$$3x+8=20$$
$$3x=20-8$$
$$3x=12$$
$$x=4, \textit{ Ans.}$$

NOTE 1: These steps can be shortened by "transposing" terms; this means that any term of an equation can be "brought" from either side to the other side, *provided its sign is reversed*. Thus, in one step, we could write:

$$5x-2x=20-8$$
then $$3x=12$$
$$x=4$$

NOTE 2: The solution of an equation should always be checked by substituting the value obtained for the letter *in the original equation;* this should yield an *identity*. Thus:

$$(5)(4)+8=(2)(4)+20$$
$$20+8=8+20$$
$$28\equiv28, \text{ Check.}$$

**Changing the Subject of Any Simple Formula in General.** The same procedure described in the foregoing paragraph applies to simple formulas as well as to simple equations.

EXAMPLE 1: Solve for $F$ the temperature formula: $C=\frac{5}{9}(F-32)$.

SOLUTION:
$$C=(\tfrac{5}{9})(F-32)$$
$$\tfrac{9}{5}C=F-32$$
$$F-32=\tfrac{9}{5}C$$
$$F=\tfrac{9}{5}C+32, \textit{ Ans.}$$

EXAMPLE 2: Solve for $R$: $C=\dfrac{E}{R+nr}$

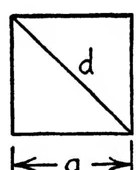
SOLUTION:
$$C=\frac{E}{R+nr}$$
$$C\,(R+nr)=E$$
$$CR+Cnr=E$$
$$CR=E-Cnr$$
$$R=\frac{E-Cnr}{C},\ \textit{Ans.}$$

*Exercise 38.*

Solve the following equations:

1. $8.5x+5=22$; find $x$.
2. $28-4y=3(2y-4)$; find $y$.
3. $n=p+kr$; solve for $r$.
4. $C=4p+q$; solve for $p$.
5. $P=2(l+w)$; solve for $w$.
6. $V=E+Ir$; solve for $I$.
7. $T=\dfrac{12(D-d)}{L}$; solve for $d$.
8. $V=v_0+gt$; solve for $t$.
9. $A=\frac{1}{2}h(B+b)$; solve for $B$.
10. $A=P+PRT$; solve for $R$.
11. $l=a+(n-1)d$; solve for $d$.
12. $C=\dfrac{N-n}{2P}$; solve for $n$.
13. $D=\dfrac{2\pi R}{L}$; solve for $R$.
14. $N=\dfrac{S-W}{W}$; solve for $S$; also for $W$.
15. $F=\dfrac{Wv^2}{gR}$; solve for $W$; for $R$.
16. $Q=\dfrac{w(H-h)}{t_1-t_0}$; solve for $H$.

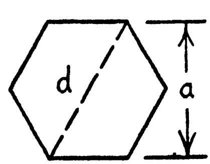
**Practical Use of Formulas in the Shop.** It should be pointed out that formulas of all kinds are constantly used in the various trades, in the shop, and in industrial work. For illustrative purposes as well as for reference and for self-practice a number of typical formulas are given below; these and many others will be found from time to time throughout the rest of the book.




(A) *Nuts and Bolts.*

(1) Diameter of blank for square bolt:
$$d=1.414a$$

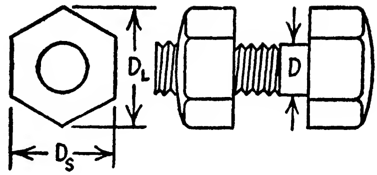
(2) Diameter of blank for hexagonal bolt:
$$d=1.155a$$



(3) Bolt and nut dimensions:

$D_S = \frac{3}{2}\,D + \frac{1}{8}$

$D_L = \frac{7}{4}\,D + \frac{1}{8}$

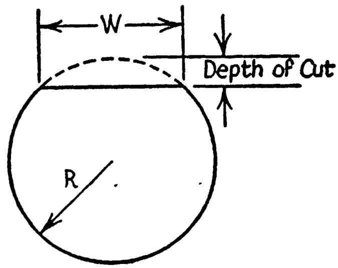



**(B)** *Depth of Cut.*

(4) When milling flats on round stock:

Depth $= R - \sqrt{R^2 - \frac{1}{4}W^2}$,

where $R$ = radius of round bar, and $W$ = width of flat surface.

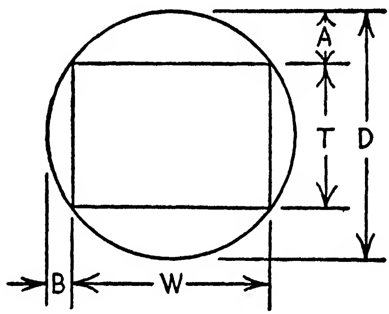



(5) When machining round stock with rectangular bars:

$$\text{Depth } A = \frac{D - T}{2},$$

$$\text{Depth } B = \frac{D - W}{2},$$

where $D$ = diameter of round stock, $W$ = width of rectangular bar, $T$ = thickness of rectangular bar, and $A$ and $B$ = depths of cut.



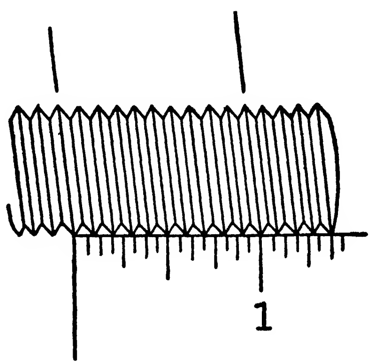

(C) *Screw Threads.*

(6) Pitch of screw ($P$) and number of threads per inch ($N$):

$$P = \frac{1}{N};\ N = \frac{1}{P}$$

(7) Depth ($d$) of sharp V-thread:

$$d = .866p = \frac{.866}{N}$$




Screw Thread : 10 Threads per inch

(8) Depth of American National thread:

$$d = .6495p = \frac{.6495}{N}$$

(9) Tap drill for sharp V-thread:

$$\text{Tap drill size } S = T - \frac{1.733}{N}$$

(10) Tap drill for American National thread:

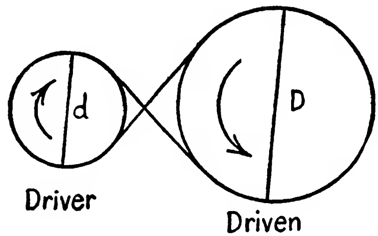
$$\text{Tap drill size } S = D - \frac{1}{N}$$

(D) *Pulley and Gear Speeds.*

(11) Speeds of belt-driven pulleys ($s$, $S$=speed, r.p.m.):

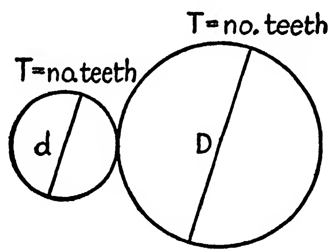
$$s = \frac{D \times S}{d}$$

$$d = \frac{D \times S}{s}$$




(12) Speed of gears in mesh ($t$, $T$= no. of teeth):

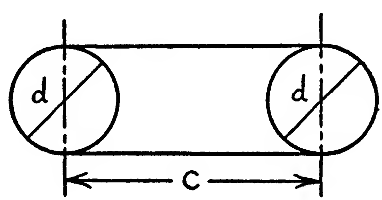
$$S = \frac{t \times s}{T}$$




(E) *Belting.*

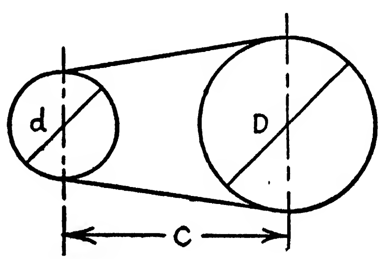
(13) Length of open belt (equal pulleys):

$$L = \pi d + 2c$$



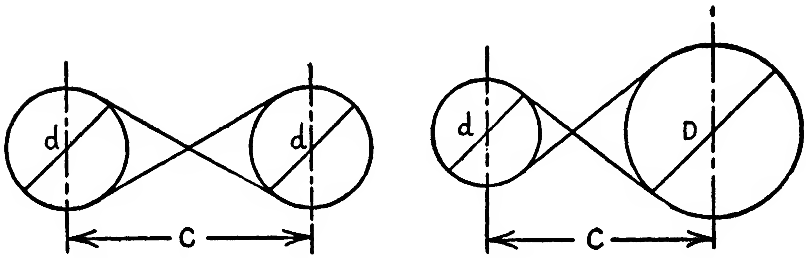

(14) Length of open belt (unequal pulleys):

$$L = \frac{\pi}{2}(D+d) + 2\sqrt{c^2 + (R-r)^2}$$



(15) Length of crossed belt, (equal or unequal pulleys):

$$L=\frac{\pi}{2}(D+d)+2\sqrt{c^2+(R+r)^2}$$




(16) Horsepower transmitted by belts:

$$\text{H.P.}=\frac{S\times W\times T}{33{,}000}, \text{ where}$$

$S$ =speed of belt (ft./min.)
$W$=width of belt (inches)
$T$ =tension in belt (40 or 70 lb.)

(F) *Energy and Power.*

(17) Work=Force×Time:

$$W=FT$$

(18) Power=Work÷Time:

$$P=\frac{W}{T}$$

(19) Horsepower:

$$\text{H.P.}=\frac{\text{ft.-lb. per min.}}{33{,}000}=\frac{\text{ft.-lb. per sec.}}{550}$$

(20) Kinetic energy of moving object:

$$\text{K.E.}=\frac{wv^2}{2g}, \text{ where}$$

$w$=weight (lb.)
$v$ =velocity (ft. per min.)
$g$ =acceleration of gravity (32 ft./sec./sec.)

(21) Indicated horsepower of steam engine:

$$\text{I.H.P.}=\frac{PLAN}{33{,}000}, \text{ where}$$

$P$ =mean effective pressure (lb./sq. in.)
$L$ =length of stroke (ft.)
$A$ =area of piston (sq. in.)
$N$ =no. of strokes (2×r.p.m.)

(22) Horsepower rating of gas engine:

$$\text{H.P.}=\frac{D^2N}{2.5}, \text{ where}$$

$D$=diameter of cylinders (in.)
$N$=number of cylinders
(assumed piston speed=1000 ft. per min.)

## 9. SQUARE ROOT

**Use of Square Rout.** For some reason or other, learning to find the square root of a number seems to be a stumbling block for most folks. As Stephen Leacock puts it, square root is as "obdurate as a hardwood stump in a pasture—nothing but years of effort can extract it. You can't hurry the process." Yet there is nothing really difficult about it. And it is a very practical matter, for, as we shall soon see, many problems in mensuration require the determination of the square root of a number. Finding a square root is simply the opposite of squaring a number; i.e., given the product of any number multiplied by itself, to find the original number. Several methods for extracting the square root of a number are available:

1. By inspection and approximation.
2. By using a table.
3. By the algebraic rule.
4. By means of logarithms.
5. By using the slide rule.

The last two methods will be explained in subsequent sections of the present chapter; the first three will now be discussed. Which method you use will depend largely upon the accuracy required and the availability of tables or a slide rule.

**Approximation Methods.** A fundamental method based upon successive trials and approximations is the following; it is somewhat laborious and therefore not particularly convenient, but will be found useful if not required too frequently, and if no other method is available. It has the advantage, however, of being simple to understand. Suppose we wish to find the square root of 44. Obviously, the value of $\sqrt{44}$ must be greater than 6, but less than 7, since 44 lies between 36 ($=6^2$) and 49 ($=7^2$). Suppose we *guess* the value to be 6.6, since 44 is somewhat closer to 49 than it is to 36. Now by actual multiplication we find that $(6.6)^2=$ 43.56, which is already quite close to 44; in fact, so close that it is safe to say that, correct to the nearest tenth, the value of $\sqrt{44}=6.6$ (the actual value=6.633). To make sure, we find by multiplication once more that $(6.7)^2=44.89$, which is too large by more than 43.56, is too small;

hence 6.6 is correct to the nearest tenth. If we wish to find the value correct to the nearest hundredth, we would continue the trial guesses and approximations until we were certain of the second decimal place. To be sure, the labor becomes a bit cumbersome; fortunately, for many purposes the result to one decimal place suffices. For greater accuracy, other methods are frankly superior. If the original guess of 6.6 had been less fortunate, say 6.5 or 6.7, the amount of trial multiplication needed to determine the first decimal place is not prohibitive. This method is not at all suited, obviously, for three-place figures or greater, e.g., to find $\sqrt{273}$ or $\sqrt{4136}$.

Another approximation method is based on the algebraic formula:

$$\sqrt{a^2+b}=a+\frac{b}{2a};$$

applying it to the same problem, we regard $\sqrt{44}$ as equal to $\sqrt{36+8}$, so that $a=6$ and $b=8$. Therefore $\sqrt{44}=\sqrt{6+8/12}=6\tfrac{2}{3}=6.67$, which is also fairly close, although when "rounded off" it would give 6.7 for the answer. This method is fairly useful for numbers somewhat larger than two-place figures. It has the advantage, moreover, of not requiring a number of tedious multiplications.

**Using a Table of Square Roots.** The values of squares and square roots of many numbers have been carefully worked out and are usually available in the form of *tables* in most handbooks and reference manuals. When such tables are accessible, their use is doubtless the most convenient method of finding the desired square root, unless, of course, it is required to find the value to more decimal places than given in the table; in that case, either the "algebraic" method or logarithms should be used. A convenient table of square roots is given on pages 89–93.

Using this table we find the $\sqrt{44}$ directly to be 6.633, as already mentioned. However, it should be noted that the table can be used for many more numbers than appear in the column under the heading "N". Thus $\sqrt{44.8}=6.693$; also the $\sqrt{4400}=66.33$; and $\sqrt{4480}=66.93$. In other words, moving the decimal point *two places* in N means that it must be moved *one place* in the root. Furthermore, to find $\sqrt{4.4}$, we look on page 89 under "N" for 4.4 and find that the required value of $\sqrt{4.4}=2.098$; similarly, $\sqrt{4.45}=2.110$; $\sqrt{445}=21.10$; and $\sqrt{440}=20.98$, while $\sqrt{44{,}000}=209.8$.

**Square Root by the Algebraic Rule.** This method is a "rule of thumb" procedure based on the algebraic relation that $(a+b)^2=a^2+2ab+b^2$. Without stopping to explain the theory in full, we shall simply illustrate the procedure.

**Example 1:** Find the $\sqrt{44}$.

**Solution:**

```
              44.000000 )6.633+, Ans.
              36
            ┌──────
 12   6     │8 00
            │7 56
            └──────────
 132  3        │4400
               │3969
               ┌──────
 1326 3        │43100
               │39789
```

Point off in blocks of two, beginning at the decimal point.
Largest square in $44=36; \sqrt{36}=6$, first digit in root.
Double $6=12$; "trial divisor" $=126$. Multiplying 126 by $6=756$.
Double root so far obtained; twice $66=132$. Second "trial divisor" $=1323$. Multiply 1323 by $3=3969$.
Double root so far obtained; twice $663=1326$. Third "trial divisor" $=13263$; $13263 \times 3=39789$.
And so on, to as many decimal places as desired.

**Example 2:** Find the value of $\sqrt{79{,}328}$ correct to the nearest hundredth.

**Solution:** 79328.0000 )281.65+, *Ans.*

```
            4
 48       │3 93
          │3 84
          └──────────
 561          │9 28
              │5 61
              ┌──────────
 5626         │3 67 00
              │3 37 56
              └──────────
 56325           │29 44 00
                 │28 16 25
```

It is clear that to find the square root of a number of 4 or 5 places, or more, neither the approximation method nor the table is of very much help; for that matter, neither is the slide rule. Only an extensive table of logarithms would do. Hence the "algebraic" method is quite useful, even though it may be slightly annoying at first.

### *Exercise 39.*

Find, to the nearest tenth, the square root of each of the following by an approximation method; then check your result by means of the table:

| | | | |
|---|---|---|---|
| 1. 18 | 4. 41 | 7. 55 | 10. 150 |
| 2. 30 | 5. 60 | 8. 74 | 11. 172 |
| 3. 12 | 6. 21 | 9. 129 | 12. 200 |

Find, to the nearest tenth, the square root of each of the following by using the algebraic rule; check your result, by referring to the table:

| | | | |
|---|---|---|---|
| 13. 86 | 15. 6.25 | 17. 567 | 19. 69.43 |
| 14. 54.8 | 16. 382 | 18. 2933 | 20. 426.5 |

**Evaluation of Formulas Involving Square Root.** It so happens that formulas in mensuration, shop work and science problems frequently involve the square root of a constant or of one of the variables. If we wish to evaluate such a formula, it is useful to be able to find the numerical value of a square root quickly and easily; the following problems will afford practice in such computation.

*Exercise 40.*

1. Find the value of $d$ in the formula $d=\sqrt{2hr}$ when $r=4.2$ and $h=6.5$.
2. Find the value of $D$ in the formula $D=\sqrt{\frac{1}{3}RS}$ when $R=28$ and $S=6\frac{1}{4}$.
3. Find the value of $t$ in the formula $t=\frac{44}{7}\sqrt{\frac{l}{g}}$ when $l=44.1$ and $g=9.80$
4. Find the value of $A$ when $S=8.6$ in the formula $A=\frac{S^2}{4}\sqrt{3}$.
5. If $s=650$ and $g=32$, find the value of $t$ to the nearest tenth in the formula $t=\sqrt{\frac{2s}{g}}$
6. Find $b$ to the nearest hundredth from the formula $b=\sqrt{c^2+a^2}$ when $c=10.4$ and $a=4.2$.
7. The volume of a frustum of a pyramid is given by $V=\frac{1}{3}h(B_1+B_2+\sqrt{B_1B_2})$. Find $V$ if $B_1=34$, $B_2=22$, and $h=6$.
8. The maximum distance at which an object on the earth may be seen from a point above the earth's surface is given by the formula $D=1.22\sqrt{E}$, where $D$ is the distance of the object in miles and $E$ is the elevation of the observer in feet. Find $D$ when $E=840$ ft.
9. The "effective" area of a smokestack is expressed by the formula $E=A-\frac{3}{5}\sqrt{A}$, where $A$ is its measured area in square feet. Find $E$ when $A=44$ sq. ft.
10. The amount of sag ($d$) in a rope or wire suspended between two points is given by

$$d=\sqrt{\frac{3l(L-l)}{8}},$$

where $l$=length of the rope in feet when taut, $L$=its actual length in feet, and $d$ equals the maximum sag in inches. Find $d$ when $L=50$ and $l=45$.

**Solving Equations Containing Radicals.** Formulas and equations frequently involve radicals, particularly the square root of a quantity, whether a constant or a radical; or they contain terms that are raised to the second power. The procedure in such cases is illustrated by the following.

EXAMPLE 1: If $A=\frac{\pi d^2}{4}$, find $d$.

SOLUTION: $4A=\pi d^2$

$$d^2=\frac{4A}{\pi}$$

$$d=\sqrt{\frac{4A}{\pi}}=2\sqrt{\frac{A}{\pi}}, \textit{ Ans.}$$

EXAMPLE 2: Solve for $V$: $F=\frac{wV^2}{gr}$

SOLUTION: $grF=wV^2$

$$V^2=\frac{grF}{w}$$

$$V=\sqrt{\frac{grF}{w}}, \textit{ Ans.}$$

EXAMPLE 3: If $a^2=b^2+c^2-2cp$, find $b$.

SOLUTION: $b^2=a^2-c^2+2cp$

$$b=\sqrt{a^2-c^2+2cp}, \textit{ Ans.}$$

EXAMPLE 4: In the formula $t=\sqrt{\frac{2s}{g}}$, solve for $s$.

SOLUTION: $t^2=\frac{2s}{g}$

$$gt^2=2s$$

$$s=\frac{gt^2}{2}, \textit{ Ans.}$$

EXAMPLE 5: Solve for $r$: $p=\frac{a}{2\pi\sqrt{nr}}$

SOLUTION: $p^2=\frac{a^2}{4\pi^2 nr}$

$$4\pi^2 nrp^2=a^2$$

$$r=\frac{a^2}{4n\pi^2p^2}, \textit{ Ans.}$$

### *Exercise 41.*

1. The kinetic energy of a body in motion is given by the formula $E=\frac{1}{2}mv^2$, where $m$ is its mass and $v$ its velocity. Solve for $v$.
2. The velocity in feet per second of a freely falling body at $t$ seconds after it began falling is found from the formula $v=\frac{1}{2}gt^2$, where $g=$ the acceleration of gravity (32 ft. per sec.). Solve for $t$.

3. If $m$ is the maximum "visibility" in miles from an elevation above the earth of $h$ feet, we have $m=\sqrt{\frac{3h}{2}}$. Solve for $h$.
4. If $h$ is the altitude of an equilateral triangle with side $s$, then $h^2=\frac{3s^2}{4}$. Solve for $h$.
5. Solve the formula $A=\frac{s^2}{4}\sqrt{3}$ for $s$.
6. The volume of a circular cone is given by $V=\frac{1}{3}\pi R^2h$. Solve for $R$.
7. In dealing with a radio circuit, the formula $f=\frac{1}{2\pi\sqrt{LC}}$ is used. Solve for $C$.
8. In computing the strength of a certain beam, the formula $\frac{I}{C^2}=\frac{\pi ab}{6}$ is needed. Solve for $C$.
9. The formula $P=\frac{Wv^2}{550\times 2g}$ is used in the determining horsepower. Solve for $v$.
10. A formula of electrical engineering is $C=\frac{E}{\sqrt{R^2+(2\pi nL)^2}}$. Find $R$.
11. The area in sq. ft. of the cross-section of a smoke stack required to carry off the smoke is given by $A=\frac{.6P}{\sqrt{h}}$, where $P$=no. of lb. of coal burned per hr., and $h$=height of stack in ft. Solve for $h$.
12. In Ex. 11, find $A$ if $h=75$ ft. and $P=550$ lb.

**Solving a Simple Quadratic Equation.** A simple quadratic formula or equation is one which contains second-degree (squared) terms as well as first-degree terms, or only squared terms, involving one of the variables only, and no higher powers of any other variable; examples of such quadratic equations are:

| | |
|---|---|
| (1) $y=3x^2$ | (4) $H=I^2R$ |
| (2) $y=5x^2+3x-10$ | (5) $V=at+\frac{1}{2}at^2$ |
| (3) $E=\frac{1}{2}mv^2$ | (6) $V=\frac{1}{3}\pi R^2h$ |

The general type of a simple quadratic function may be written as

$$y=ax^2+bx+c,$$

and the general form of a simple quadratic equation is

$$ax^2+bx+c=o,$$

where $a$, $b$ and $c$ are the numerical coefficients. Thus, if the equation to be solved is $3x^2+5x-7=0$, then $a=3$, $b=5$, and $c=-7$.

The solution of such an equation is found by using the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Without explaining how this formula is derived, we shall illustrate its use in solving quadratic equations.

EXAMPLE 1: Solve for $x$: $3x^2+5x-7=0$.

SOLUTION:
$$x=\frac{-5\pm\sqrt{25-(4)(3)(-7)}}{(2)(3)},$$

$$\text{or } x=\frac{-5\pm\sqrt{25+84}}{6},$$

$$x=\frac{-5\pm\sqrt{109}}{6}=\frac{-5\pm10.44}{6}.$$

$$\left.\begin{array}{l}\text{Thus } x\dfrac{-5+10.44}{6}=\dfrac{5.44}{6}=+.91; \\ \text{and also } x\dfrac{-5-10.44}{6}=\dfrac{-15.44}{6}=-2.57\end{array}\right\} Ans.$$

It should be noted from the above that there are *two values* for $x$; each of them satisfies the original equation. Every quadratic equation thus has *two roots,* although in the case of formulas, one of them might have no *practical* significance in the relation represented by the formula.

EXAMPLE 2: Solve to the nearest hundredth for $t$:

$$5t^2-10t+2=0$$

SOLUTION: In this case $a=5$, $b=-10$, and $c=2$.

$$\text{Then } t=\frac{+10\pm\sqrt{100-(4)(5)(2)}}{(2)(5)}$$

$$t=\frac{10\pm\sqrt{20}}{10}=\frac{10\pm4.472}{10}$$

$$t=+1.45 \text{ and } +.55, \ Ans.$$

NOTE: Before substituting in the formula, always transpose *all* terms to the left side of the equation; otherwise the signs of $a$, $b$ and $c$ will be incorrect.

## *Exercise 42.*

Solve the following, finding the roots correct to the nearest tenth:

1. $3x^2-4x-11=0$
2. $x^2-30=10x$
3. $y^2+y=14$
4. $4k^2-k=8$
5. $2p^2-2=7p$
6. $32t+16t^2=240$

## SQUARE ROOTS OF NUMBERS*

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.0 | 1.000 | 1.005 | 1.010 | 1.015 | 1.020 | 1.025 | 1.030 | 1.034 | 1.039 | 1.044 |
| 1 | 1.049 | 1.054 | 1.058 | 1.063 | 1.068 | 1.072 | 1.077 | 1.082 | 1.086 | 1.091 |
| 2 | 1.095 | 1.100 | 1.105 | 1.109 | 1.114 | 1.118 | 1.122 | 1.127 | 1.131 | 1.136 |
| 3 | 1.140 | 1.145 | 1.149 | 1.153 | 1.158 | 1.162 | 1.166 | 1.170 | 1.175 | 1.179 |
| 4 | 1.183 | 1.187 | 1.192 | 1.196 | 1.200 | 1.204 | 1.208 | 1.212 | 1.217 | 1.221 |
| 1.5 | 1.225 | 1.229 | 1.233 | 1.237 | 1.241 | 1.245 | 1.249 | 1.253 | 1.257 | 1.261 |
| 6 | 1.265 | 1.269 | 1.273 | 1.277 | 1.281 | 1.285 | 1.288 | 1.292 | 1.296 | 1.300 |
| 7 | 1.304 | 1.308 | 1.311 | 1.315 | 1.319 | 1.323 | 1.327 | 1.330 | 1.334 | 1.338 |
| 8 | 1.342 | 1.345 | 1.349 | 1.353 | 1.356 | 1.360 | 1.364 | 1.367 | 1.371 | 1.375 |
| 9 | 1.378 | 1.382 | 1.386 | 1.389 | 1.393 | 1.396 | 1.400 | 1.404 | 1.407 | 1.411 |
| 2.0 | 1.414 | 1.418 | 1.421 | 1.425 | 1.428 | 1.432 | 1.435 | 1.439 | 1.442 | 1.446 |
| 1 | 1.449 | 1.453 | 1.456 | 1.459 | 1.463 | 1.466 | 1.470 | 1.473 | 1.476 | 1.480 |
| 2 | 1.483 | 1.487 | 1.490 | 1.493 | 1.497 | 1.500 | 1.503 | 1.507 | 1.510 | 1.513 |
| 3 | 1.517 | 1.520 | 1.523 | 1.526 | 1.530 | 1.533 | 1.536 | 1.539 | 1.543 | 1.546 |
| 4 | 1.549 | 1.552 | 1.556 | 1.559 | 1.562 | 1.565 | 1.568 | 1.572 | 1.575 | 1.578 |
| 2.5 | 1.581 | 1.584 | 1.587 | 1.591 | 1.594 | 1.597 | 1.600 | 1.603 | 1.606 | 1.609 |
| 6 | 1.612 | 1.616 | 1.619 | 1.622 | 1.625 | 1.628 | 1.631 | 1.634 | 1.637 | 1.640 |
| 7 | 1.643 | 1.646 | 1.649 | 1.652 | 1.655 | 1.658 | 1.661 | 1.664 | 1.667 | 1.670 |
| 8 | 1.673 | 1.676 | 1.679 | 1.682 | 1.685 | 1.688 | 1.691 | 1.694 | 1.697 | 1.700 |
| 9 | 1.703 | 1.706 | 1.709 | 1.712 | 1.715 | 1.718 | 1.720 | 1.723 | 1.726 | 1.729 |
| 3.0 | 1.732 | 1.735 | 1.738 | 1.741 | 1.744 | 1.746 | 1.749 | 1.752 | 1.755 | 1.758 |
| 1 | 1.761 | 1.764 | 1.766 | 1.769 | 1.772 | 1.775 | 1.778 | 1.780 | 1.783 | 1.786 |
| 2 | 1.789 | 1.792 | 1.794 | 1.797 | 1.800 | 1.803 | 1.806 | 1.808 | 1.811 | 1.814 |
| 3 | 1.817 | 1.819 | 1.822 | 1.825 | 1.828 | 1.830 | 1.833 | 1.836 | 1.838 | 1.841 |
| 4 | 1.844 | 1.847 | 1.849 | 1.852 | 1.855 | 1.857 | 1.860 | 1.863 | 1.865 | 1.868 |
| 3.5 | 1.871 | 1.873 | 1.876 | 1.879 | 1.881 | 1.884 | 1.887 | 1.889 | 1.892 | 1.895 |
| 6 | 1.897 | 1.900 | 1.903 | 1.905 | 1.908 | 1.910 | 1.913 | 1.916 | 1.918 | 1.921 |
| 7 | 1.924 | 1.926 | 1.929 | 1.931 | 1.934 | 1.936 | 1.939 | 1.942 | 1.944 | 1.947 |
| 8 | 1.949 | 1.952 | 1.954 | 1.957 | 1.960 | 1.962 | 1.965 | 1.967 | 1.970 | 1.972 |
| 9 | 1.975 | 1.977 | 1.980 | 1.982 | 1.985 | 1.987 | 1.990 | 1.992 | 1.995 | 1.997 |
| 4.0 | 2.000 | 2.002 | 2.005 | 2.007 | 2.010 | 2.012 | 2.015 | 2.017 | 2.020 | 2.022 |
| 1 | 2.025 | 2.027 | 2.030 | 2.032 | 2.035 | 2.037 | 2.040 | 2.042 | 2.045 | 2.047 |
| 2 | 2.049 | 2.052 | 2.054 | 2.057 | 2.059 | 2.062 | 2.064 | 2.066 | 2.069 | 2.071 |
| 3 | 2.074 | 2.076 | 2.078 | 2.081 | 2.083 | 2.086 | 2.088 | 2.090 | 2.093 | 2.095 |
| 4 | 2.098 | 2.100 | 2.102 | 2.105 | 2.107 | 2.110 | 2.112 | 2.114 | 2.117 | 2.119 |

*From Mechanical Engineers' Handbook, by Lionel S. Marks (1941). Courtesy of the McGraw-Hill Book Co.

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 4.5 | 2.121 | 2.124 | 2.126 | 2.128 | 2.131 | 2.133 | 2.135 | 2.138 | 2.140 | 2.142 |
| 6 | 2.145 | 2.147 | 2.149 | 2.152 | 2.154 | 2.156 | 2.159 | 2.161 | 2.163 | 2.166 |
| 7 | 2.168 | 2.170 | 2.173 | 2.175 | 2.177 | 2.179 | 2.182 | 2.184 | 2.186 | 2.189 |
| 8 | 2.191 | 2.193 | 2.195 | 2.198 | 2.200 | 2.202 | 2.205 | 2.207 | 2.209 | 2.211 |
| 9 | 2.214 | 2.216 | 2.218 | 2.220 | 2.223 | 2.225 | 2.227 | 2.229 | 2.232 | 2.234 |
| 5.0 | 2.236 | 2.238 | 2.241 | 2.243 | 2.245 | 2.247 | 2.249 | 2.252 | 2.254 | 2.256 |
| 1 | 2.258 | 2.261 | 2.263 | 2.265 | 2.267 | 2.269 | 2.272 | 2.274 | 2.276 | 2.278 |
| 2 | 2.280 | 2.283 | 2.285 | 2.287 | 2.289 | 2.291 | 2.293 | 2.296 | 2.298 | 2.300 |
| 3 | 2.302 | 2.304 | 2.307 | 2.309 | 2.311 | 2.313 | 2.315 | 2.317 | 2.319 | 2.322 |
| 4 | 2.324 | 2.326 | 2.328 | 2.330 | 2.332 | 2.335 | 2.337 | 2.339 | 2.341 | 2.343 |
| 5.5 | 2.345 | 2.347 | 2.349 | 2.352 | 2.354 | 2.356 | 2.358 | 2.360 | 2.362 | 2.364 |
| 6 | 2.366 | 2.369 | 2.371 | 2.373 | 2.375 | 2.377 | 2.379 | 2.381 | 2.383 | 2.385 |
| 7 | 2.387 | 2.390 | 2.392 | 2.394 | 2.396 | 2.398 | 2.400 | 2.402 | 2.404 | 2.406 |
| 8 | 2.408 | 2.410 | 2.412 | 2.415 | 2.417 | 2.419 | 2.421 | 2.423 | 2.425 | 2.427 |
| 9 | 2.429 | 2.431 | 2.433 | 2.435 | 2.437 | 2.439 | 2.441 | 2.443 | 2.445 | 2.447 |
| 6.0 | 2.449 | 2.452 | 2.454 | 2.456 | 2.458 | 2.460 | 2.462 | 2.464 | 2.466 | 2.468 |
| 1 | 2.470 | 2.472 | 2.474 | 2.476 | 2.478 | 2.480 | 2.482 | 2.484 | 2.486 | 2.488 |
| 2 | 2.490 | 2.492 | 2.494 | 2.496 | 2.498 | 2.500 | 2.502 | 2.504 | 2.506 | 2.508 |
| 3 | 2.510 | 2.512 | 2.514 | 2.516 | 2.518 | 2.520 | 2.522 | 2.524 | 2.526 | 2.528 |
| 4 | 2.530 | 2.532 | 2.534 | 2.536 | 2.538 | 2.540 | 2.542 | 2.544 | 2.546 | 2.548 |
| 6.5 | 2.550 | 2.551 | 2.553 | 2.555 | 2.557 | 2.559 | 2.561 | 2.563 | 2.565 | 2.567 |
| 6 | 2.569 | 2.571 | 2.573 | 2.575 | 2.577 | 2.579 | 2.581 | 2.583 | 2.585 | 2.587 |
| 7 | 2.588 | 2.590 | 2.592 | 2.594 | 2.596 | 2.598 | 2.600 | 2.602 | 2.604 | 2.606 |
| 8 | 2.608 | 2.610 | 2.612 | 2.613 | 2.615 | 2.617 | 2.619 | 2.621 | 2.623 | 2.625 |
| 9 | 2.627 | 2.629 | 2.631 | 2.632 | 2.634 | 2.636 | 2.638 | 2.640 | 2.642 | 2.644 |
| 7.0 | 2.646 | 2.648 | 2.650 | 2.651 | 2.653 | 2.655 | 2.657 | 2.659 | 2.661 | 2.663 |
| 1 | 2.665 | 2.666 | 2.668 | 2.670 | 2.672 | 2.674 | 2.676 | 2.678 | 2.680 | 2.681 |
| 2 | 2.683 | 2.685 | 2.687 | 2.689 | 2.691 | 2.693 | 2.694 | 2.696 | 2.698 | 2.700 |
| 3 | 2.702 | 2.704 | 2.706 | 2.707 | 2.709 | 2.711 | 2.713 | 2.715 | 2.717 | 2.718 |
| 4 | 2.720 | 2.722 | 2.724 | 2.726 | 2.728 | 2.729 | 2.731 | 2.733 | 2.735 | 2.737 |
| 7.5 | 2.739 | 2.740 | 2.742 | 2.744 | 2.746 | 2.748 | 2.750 | 2.751 | 2.753 | 2.755 |
| 6 | 2.757 | 2.759 | 2.760 | 2.762 | 2.764 | 2.766 | 2.768 | 2.769 | 2.771 | 2.773 |
| 7 | 2.775 | 2.777 | 2.778 | 2.780 | 2.782 | 2.784 | 2.786 | 2.787 | 2.789 | 2.791 |
| 8 | 2.793 | 2.795 | 2.796 | 2.798 | 2.800 | 2.802 | 2.804 | 2.805 | 2.807 | 2.809 |
| 9 | 2.811 | 2.812 | 2.814 | 2.816 | 2.818 | 2.820 | 2.821 | 2.823 | 2.825 | 2.827 |
| 8.0 | 2.828 | 2.830 | 2.832 | 2.834 | 2.835 | 2.837 | 2.839 | 2.841 | 2.843 | 2.844 |
| 1 | 2.846 | 2.848 | 2.850 | 2.851 | 2.853 | 2.855 | 2.857 | 2.858 | 2.860 | 2.862 |
| 2 | 2.864 | 2.865 | 2.867 | 2.869 | 2.871 | 2.872 | 2.874 | 2.876 | 2.877 | 2.879 |
| 3 | 2.881 | 2.883 | 2.884 | 2.886 | 2.888 | 2.890 | 2.891 | 2.893 | 2.895 | 2.897 |
| 4 | 2.898 | 2.900 | 2.902 | 2.903 | 2.905 | 2.907 | 2.909 | 2.910 | 2.912 | 2.914 |

| *N* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 8.5 | 2.915 | 2.917 | 2.919 | 2.921 | 2.922 | 2.924 | 2.926 | 2.927 | 2.929 | 2.931 |
| 6 | 2.933 | 2.934 | 2.936 | 2.938 | 2.939 | 2.941 | 2.943 | 2.944 | 2.946 | 2.948 |
| 7 | 2.950 | 2.951 | 2.953 | 2.955 | 2.956 | 2.958 | 2.960 | 2.961 | 2.963 | 2.965 |
| 8 | 2.966 | 2.968 | 2.970 | 2.972 | 2.973 | 2.975 | 2.977 | 2.978 | 2.980 | 2.982 |
| 9 | 2.983 | 2.985 | 2.987 | 2.988 | 2.990 | 2.992 | 2.993 | 2.995 | 2.997 | 2.998 |
| 9.0 | 3.000 | 3.002 | 3.003 | 3.005 | 3.007 | 3.008 | 3.010 | 3.012 | 3.013 | 3.015 |
| 1 | 3.017 | 3.018 | 3.020 | 3.022 | 3.023 | 3.025 | 3.027 | 3.028 | 3.030 | 3.032 |
| 2 | 3.033 | 3.035 | 3.036 | 3.038 | 3.040 | 3.041 | 3.043 | 3.045 | 3.046 | 3.048 |
| 3 | 3.050 | 3.051 | 3.053 | 3.055 | 3.056 | 3.058 | 3.059 | 3.061 | 3.063 | 3.064 |
| 4 | 3.066 | 3.068 | 3.069 | 3.071 | 3.072 | 3.074 | 3.076 | 3.077 | 3.079 | 3.081 |
| 9.5 | 3.082 | 3.084 | 3.085 | 3.087 | 3.089 | 3.090 | 3.092 | 3.094 | 3.095 | 3.097 |
| 6 | 3.098 | 3.100 | 3.102 | 3.103 | 3.105 | 3.106 | 3.108 | 3.110 | 3.111 | 3.113 |
| 7 | 3.114 | 3.116 | 3.118 | 3.119 | 3.121 | 3.122 | 3.124 | 3.126 | 3.127 | 3.129 |
| 8 | 3.130 | 3.132 | 3.134 | 3.135 | 3.137 | 3.138 | 3.140 | 3.142 | 3.143 | 3.145 |
| 9 | 3.146 | 3.148 | 3.150 | 3.151 | 3.153 | 3.154 | 3.156 | 3.158 | 3.159 | 3.161 |
| 10 | 3.162 | 3.178 | 3.194 | 3.209 | 3.225 | 3.240 | 3.256 | 3.271 | 3.286 | 3.302 |
| 1 | 3.317 | 3.332 | 3.347 | 3.362 | 3.376 | 3.391 | 3.406 | 3.421 | 3.435 | 3.450 |
| 2 | 3.464 | 3.479 | 3.493 | 3.507 | 3.521 | 3.536 | 3.550 | 3.564 | 3.578 | 3.592 |
| 3 | 3.606 | 3.619 | 3.633 | 3.647 | 3.661 | 3.674 | 3.688 | 3.701 | 3.715 | 3.728 |
| 4 | 3.742 | 3.755 | 3.768 | 3.782 | 3.795 | 3.808 | 3.821 | 3.834 | 3.847 | 3.860 |
| 15 | 3.873 | 3.886 | 3.899 | 3.912 | 3.924 | 3.937 | 3.950 | 3.962 | 3.975 | 3.987 |
| 6 | 4.000 | 4.012 | 4.025 | 4.037 | 4.050 | 4.062 | 4.074 | 4.087 | 4.099 | 4.111 |
| 7 | 4.123 | 4.135 | 4.147 | 4.159 | 4.171 | 4.183 | 4.195 | 4.207 | 4.219 | 4.231 |
| 8 | 4.243 | 4.254 | 4.266 | 4.278 | 4.290 | 4.301 | 4.313 | 4.324 | 4.336 | 4.347 |
| 9 | 4.359 | 4.370 | 4.382 | 4.393 | 4.405 | 4.416 | 4.427 | 4.438 | 4.450 | 4.461 |
| 20 | 4.472 | 4.483 | 4.494 | 4.506 | 4.517 | 4.528 | 4.539 | 4.550 | 4.561 | 4.572 |
| 1 | 4.583 | 4.593 | 4.604 | 4.615 | 4.626 | 4.637 | 4.648 | 4.658 | 4.669 | 4.680 |
| 2 | 4.690 | 4.701 | 4.712 | 4.722 | 4.733 | 4.743 | 4.754 | 4.764 | 4.775 | 4.785 |
| 3 | 4.796 | 4.806 | 4.817 | 4.827 | 4.837 | 4.848 | 4.858 | 4.868 | 4.879 | 4.889 |
| 4 | 4.899 | 4.909 | 4.919 | 4.930 | 4.940 | 4.950 | 4.960 | 4.970 | 4.980 | 4.990 |
| 25 | 5.000 | 5.010 | 5.020 | 5.030 | 5.040 | 5.050 | 5.060 | 5.070 | 5.079 | 5.089 |
| 6 | 5.099 | 5.109 | 5.119 | 5.128 | 5.138 | 5.148 | 5.158 | 5.167 | 5.177 | 5.187 |
| 7 | 5.196 | 5.206 | 5.215 | 5.225 | 5.235 | 5.244 | 5.254 | 5.263 | 5.273 | 5.282 |
| 8 | 5.292 | 5.301 | 5.310 | 5.320 | 5.329 | 5.339 | 5.348 | 5.357 | 5.367 | 5.376 |
| 9 | 5.385 | 5.394 | 5.404 | 5.413 | 5.422 | 5.431 | 5.441 | 5.450 | 5.459 | 5.468 |
| 30 | 5.477 | 5.486 | 5.495 | 5.505 | 5.514 | 5.523 | 5.532 | 5.541 | 5.550 | 5.559 |
| 1 | 5.568 | 5.577 | 5.586 | 5.595 | 5.604 | 5.612 | 5.621 | 5.630 | 5.639 | 5.648 |
| 2 | 5.657 | 5.666 | 5.675 | 5.683 | 5.692 | 5.701 | 5.710 | 5.718 | 5.727 | 5.736 |
| 3 | 5.745 | 5.753 | 5.762 | 5.771 | 5.779 | 5.788 | 5.797 | 5.805 | 5.814 | 5.822 |
| 4 | 5.831 | 5.840 | 5.848 | 5.857 | 5.865 | 5.874 | 5.882 | 5.891 | 5.899 | 5.908 |

## SQUARE ROOTS (*continued*)

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 35 | 5.916 | 5.925 | 5.933 | 5.941 | 5.950 | 5.958 | 5.967 | 5.975 | 5.983 | 5.992 |
| 6 | 6.000 | 6.008 | 6.017 | 6.025 | 6.033 | 6.042 | 6.050 | 6.058 | 0.066 | 6.075 |
| 7 | 6.083 | 6.091 | 6.099 | 6.107 | 6.116 | 6.124 | 6.132 | 6.140 | 6.148 | 6.156 |
| 8 | 6.164 | 6.173 | 6.181 | 6.189 | 6.197 | 6.205 | 6.213 | 6.221 | 6.229 | 6.237 |
| 9 | 6.245 | 6.253 | 6.261 | 6.269 | 6.277 | 6.285 | 6.293 | 6.301 | 6.309 | 6.317 |
| 40 | 6.325 | 6.332 | 6.340 | 6.348 | 6.356 | 6.364 | 6.372 | 6.380 | 6.387 | 6.395 |
| 1 | 6.403 | 6.411 | 6.419 | 6.427 | 6.434 | 6.442 | 6.450 | 6.458 | 6.465 | 6.473 |
| 2 | 6.481 | 6.488 | 6.496 | 6.504 | 6.512 | 6.519 | 6.527 | 6.535 | 6.542 | 6.550 |
| 3 | 6.557 | 6.565 | 6.573 | 6.580 | 6.588 | 6.595 | 6.603 | 6.611 | 6.618 | 6.626 |
| 4 | 6.633 | 6.641 | 6.648 | 6.656 | 6.663 | 6.671 | 6.678 | 6.686 | 6.693 | 6.701 |
| 45 | 6.708 | 6.716 | 6.723 | 6.731 | 6.738 | 6.745 | 6.753 | 6.760 | 6.768 | 6.775 |
| 6 | 6.782 | 6.790 | 6.797 | 6.804 | 6.812 | 6.819 | 6.826 | 6.834 | 6.841 | 6.848 |
| 7 | 6.856 | 6.863 | 6.870 | 6.877 | 6.885 | 6.892 | 6.899 | 6.907 | 6.914 | 6.921 |
| 8 | 6.928 | 6.935 | 6.943 | 6.950 | 6.957 | 6.964 | 6.971 | 6.979 | 6.986 | 6.993 |
| 9 | 7.000 | 7.007 | 7.014 | 7.021 | 7.029 | 7.036 | 7.043 | 7.050 | 7.057 | 7.064 |
| 50 | 7.071 | 7.078 | 7.085 | 7.092 | 7.099 | 7.106 | 7.113 | 7.120 | 7.127 | 7.134 |
| 1 | 7.141 | 7.148 | 7.155 | 7.162 | 7.169 | 7.176 | 7.183 | 7.190 | 7.197 | 7.204 |
| 2 | 7.211 | 7.218 | 7.225 | 7.232 | 7.239 | 7.246 | 7.253 | 7.259 | 7.266 | 7.273 |
| 3 | 7.280 | 7.287 | 7.294 | 7.301 | 7.308 | 7.314 | 7.321 | 7.328 | 7.335 | 7.342 |
| 4 | 7.348 | 7.355 | 7.362 | 7.369 | 7.376 | 7.382 | 7.389 | 7.396 | 7.403 | 7.409 |
| 55 | 7.416 | 7.423 | 7.430 | 7.436 | 7.443 | 7.450 | 7.457 | 7.463 | 7.470 | 7.477 |
| 6 | 7.483 | 7.490 | 7.497 | 7.503 | 7.510 | 7.517 | 7.523 | 7.530 | 7.537 | 7.543 |
| 7 | 7.550 | 7.556 | 7.563 | 7.570 | 7.576 | 7.583 | 7.589 | 7.596 | 7.603 | 7.609 |
| 8 | 7.616 | 7.622 | 7.629 | 7.635 | 7.642 | 7.649 | 7.655 | 7.662 | 7.668 | 7.675 |
| 9 | 7.681 | 7.688 | 7.694 | 7.701 | 7.707 | 7.714 | 7.720 | 7.727 | 7.733 | 7.740 |
| 60 | 7.746 | 7.752 | 7.759 | 7.765 | 7.772 | 7.778 | 7.785 | 7.791 | 7.797 | 7.804 |
| 1 | 7.810 | 7.817 | 7.823 | 7.829 | 7.836 | 7.842 | 7.849 | 7.855 | 7.861 | 7.868 |
| 2 | 7.874 | 7.880 | 7.887 | 7.893 | 7.899 | 7.906 | 7.912 | 7.918 | 7.925 | 7.931 |
| 3 | 7.937 | 7.944 | 7.950 | 7.956 | 7.962 | 7.969 | 7.975 | 7.981 | 7.987 | 7.994 |
| 4 | 8.000 | 8.006 | 8.012 | 8.019 | 8.025 | 8.031 | 8.037 | 8.044 | 8.050 | 8.056 |
| 65 | 8.062 | 8.068 | 8.075 | 8.081 | 8.087 | 8.093 | 8.099 | 8.106 | 8.112 | 8.118 |
| 6 | 8.124 | 8.130 | 8.136 | 8.142 | 8.149 | 8.155 | 8.161 | 8.167 | 8.173 | 8.179 |
| 7 | 8.185 | 8.191 | 8.198 | 8.204 | 8.210 | 8.216 | 8.222 | 8.228 | 8.234 | 8.240 |
| 8 | 8.246 | 8.252 | 8.258 | 8.264 | 8.270 | 8.276 | 8.283 | 8.289 | 8.295 | 8.301 |
| 9 | 8.307 | 8.313 | 8.319 | 8.325 | 8.331 | 8.337 | 8.343 | 8.349 | 8.355 | 8.361 |
| 70 | 8.367 | 8.373 | 8.379 | 8.385 | 8.390 | 8.396 | 8.402 | 8.408 | 8.414 | 8.420 |
| 1 | 8.426 | 8.432 | 8.438 | 8.444 | 8.450 | 8.456 | 8.462 | 8.468 | 8.473 | 8.479 |
| 2 | 8.485 | 8.491 | 8.497 | 8.503 | 8.509 | 8.515 | 8.521 | 8.526 | 8.532 | 8.538 |
| 3 | 8.544 | 8.550 | 8.556 | 8.562 | 8.567 | 8.573 | 8.579 | 8.585 | 8.591 | 8.597 |
| 4 | 8.602 | 8.608 | 8.614 | 8.620 | 8.626 | 8.631 | 8.637 | 8.643 | 8.649 | 8.654 |

## SQUARE ROOTS (*continued*)

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 75 | 8.660 | 8.666 | 8.672 | 8.678 | 8.683 | 8.689 | 8.695 | 8.701 | 8.706 | 8.712 |
| 6 | 8.718 | 8.724 | 8.729 | 8.735 | 8.741 | 8.746 | 8.752 | 8.758 | 8.764 | 8.769 |
| 7 | 8.775 | 8.781 | 8.786 | 8.792 | 8.798 | 8.803 | 8.809 | 8.815 | 8.820 | 8.826 |
| 8 | 8.832 | 8.837 | 8.843 | 8.849 | 8.854 | 8.860 | 8.866 | 8.871 | 8.877 | 8.883 |
| 9 | 8.888 | 8.894 | 8.899 | 8.905 | 8.911 | 8.916 | 8.922 | 8.927 | 8.933 | 8.939 |
| 80 | 8.944 | 8.950 | 8.955 | 8.961 | 8.967 | 8.972 | 8.978 | 8.983 | 8.989 | 8.994 |
| 1 | 9.000 | 9.006 | 9.011 | 9.017 | 9.022 | 9.028 | 9.033 | 9.039 | 9.044 | 9.050 |
| 2 | 9.055 | 9.061 | 9.066 | 9.072 | 9.077 | 9.083 | 9.088 | 9.094 | 9.099 | 9.105 |
| 3 | 9.110 | 9.116 | 9.121 | 9.127 | 9.132 | 9.138 | 9.143 | 9.149 | 9.154 | 9.160 |
| 4 | 9.165 | 9.171 | 9.176 | 9.182 | 9.187 | 9.192 | 9.198 | 9.203 | 9.209 | 9.214 |
| 85 | 9.220 | 9.225 | 9.230 | 9.236 | 9.241 | 9.247 | 9.252 | 9.257 | 9.263 | 9.268 |
| 6 | 9.274 | 9.279 | 9.284 | 9.290 | 9.295 | 9.301 | 9.306 | 9.311 | 9.317 | 9.322 |
| 7 | 9.327 | 9.333 | 9.338 | 9.343 | 9.349 | 9.354 | 9.359 | 9.365 | 9.370 | 9.375 |
| 8 | 9.381 | 9.386 | 9.391 | 9.397 | 9.402 | 9.407 | 9.413 | 9.418 | 9.423 | 9.429 |
| 9 | 9.434 | 9.439 | 9.445 | 9.450 | 9.455 | 9.460 | 9.466 | 9.471 | 9.476 | 9.482 |
| 90 | 9.487 | 9.492 | 9.497 | 9.503 | 9.508 | 9.513 | 9.518 | 9.524 | 9.529 | 9.534 |
| 1 | 9.539 | 9.545 | 9.550 | 9.555 | 9.560 | 9.566 | 9.571 | 9.576 | 9.581 | 9.586 |
| 2 | 9.592 | 9.597 | 9.602 | 9.607 | 9.612 | 9.618 | 9.623 | 9.628 | 9.633 | 9.638 |
| 3 | 9.644 | 9.649 | 9.654 | 9.659 | 9.664 | 9.670 | 9.675 | 9.680 | 9.685 | 9.690 |
| 4 | 9.695 | 9.701 | 9.706 | 9.711 | 9.716 | 9.721 | 9.726 | 9.731 | 9.737 | 9.742 |
| 95 | 9.747 | 9.752 | 9.757 | 9.762 | 9.767 | 9.772 | 9.778 | 9.783 | 9.788 | 9.793 |
| 6 | 9.798 | 9.803 | 9.808 | 9.813 | 9.818 | 9.823 | 9.829 | 9.834 | 9.839 | 9.844 |
| 7 | 9.849 | 9.854 | 9.859 | 9.864 | 9.869 | 9.874 | 9.879 | 9.884 | 9.889 | 9.894 |
| 8 | 9.899 | 9.905 | 9.910 | 9.915 | 9.920 | 9.925 | 9.930 | 9.935 | 9.940 | 9.945 |
| 9 | 9.950 | 9.955 | 9.960 | 9.965 | 9.970 | 9.975 | 9.980 | 9.985 | 9.990 | 9.995 |

## 10. VARIATION, DEPENDENCE AND GRAPHS

**Another Use for the Formula.** By studying a formula carefully it is possible to tell how changes in the value of one of the quantities will affect the values of another quantity. Thus we note that in the formula

$$D=\frac{M}{V},$$

(1) if $V$ remains constant and $M$ is doubled, then $D$ is doubled.
(2) if $M$ remains constant and $V$ is doubled, $D$ becomes ½ as large.
(3) if $M$ and $V$ are both doubled, then $D$ remains the same.
(4) if $M$ is doubled and $V$ is halved, then $D$ becomes 4 times as large.

Or again, consider the formula $A=6e^2$;
(1) if $e$ is doubled, $A$ becomes 4 times as large.
(2) if $e$ is halved, $A$ becomes ¼ as large.
(3) if $e$ is multiplied by 3, $A$ becomes 9 times as large.
(4) if $e$ is divided by 4, $A$ becomes $1/16$ as large.
(5) if $e$ is multiplied by 10, $A$ becomes 100 times as large.

*Exercise 43.*

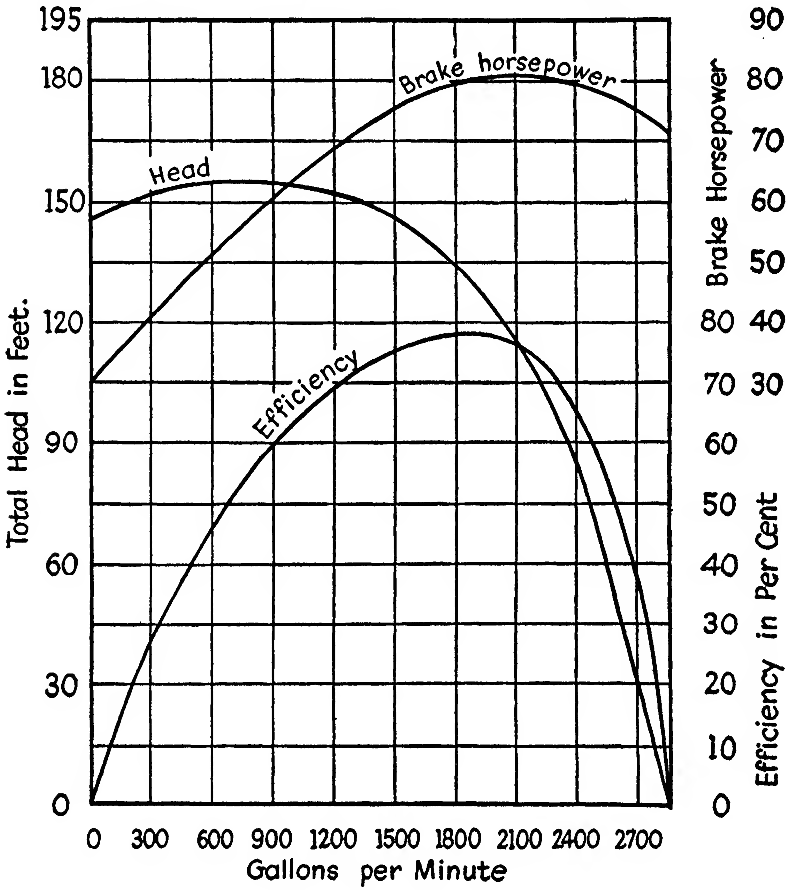
1. In the formula $D=RT$, how is $D$ changed when
   (a) $R$ is doubled and $T$ remains constant.
   (b) $T$ is halved and $R$ remains constant.
   (c) $R$ is divided by 10 and $T$ remains constant.
   (d) $T$ is tripled and $R$ is halved.
   (e) $R$ is divided by 4 and $T$ is divided by 2.

2. In the formula $A=\frac{1}{2}bh$, how is $A$ affected when
   (a) $b$ and $h$ are both tripled.
   (b) $h$ remains constant and $b$ is divided by 6.
   (c) $b$ is doubled and $h$ is halved.
   (d) $h$ is multiplied by 4 and $b$ remains constant.

3. In the formula $PV=k$, if $k$ is always constant, what change takes place when
   (a) $P$ increases.
   (b) $V$ increases.
   (c) $V$ is halved.
   (d) $P$ is tripled.
   (e) $V$ is multiplied by 5.

4. In the formula $A=\pi R^2$, what happens to $A$ when
   (a) $R$ is doubled.
   (b) $R$ is divided by 3.
   (c) $R$ is multiplied by 5.
   (d) $R$ is divided by 10.

**What a Formula Does.** By this time it will be seen that a formula has a number of advantages over the verbal statement of the relationship between two or more quantities. It is briefer and simpler; it emphasizes the nature of the relationship; it enables us to compute the value of one of the variables when specific values of the other variables are known; it permits the relationship to be restated so that any particular variable may be expressed or "described" in terms of the other variables, which is generally a convenient device. In short, a formula is a powerful mathematical tool, since it implicitly represents *all the infinite sets of corresponding values of the variables,* all at once, so to speak.




Characteristic curves of a single-stage centrifugal pump operating at a constant speed of 1,200 r.p.m.

**A Graph also Shows Dependence.** Consider the three graphs shown in the accompanying figure. Study the curve labeled "efficiency"; referring to the horizontal scale of "gallons per minute" and the vertical scale (at the lower right) of "efficiency in per cent," we can read directly from the efficiency at any particular load, or the load at any particular efficiency. Thus when pumping 2700 gal. per minute the efficiency is 40%; when the efficiency is 60%, it may be pumping a little over 900, or about 2500 gal. per minute (the 60% line crosses the curve in *two places*). Also the maximum (greatest) efficiency is reached when pumping a little under 1950 gal. per minute. And in the same way, the other two curves might be studied and analyzed. This could not be done conveniently with a formula.

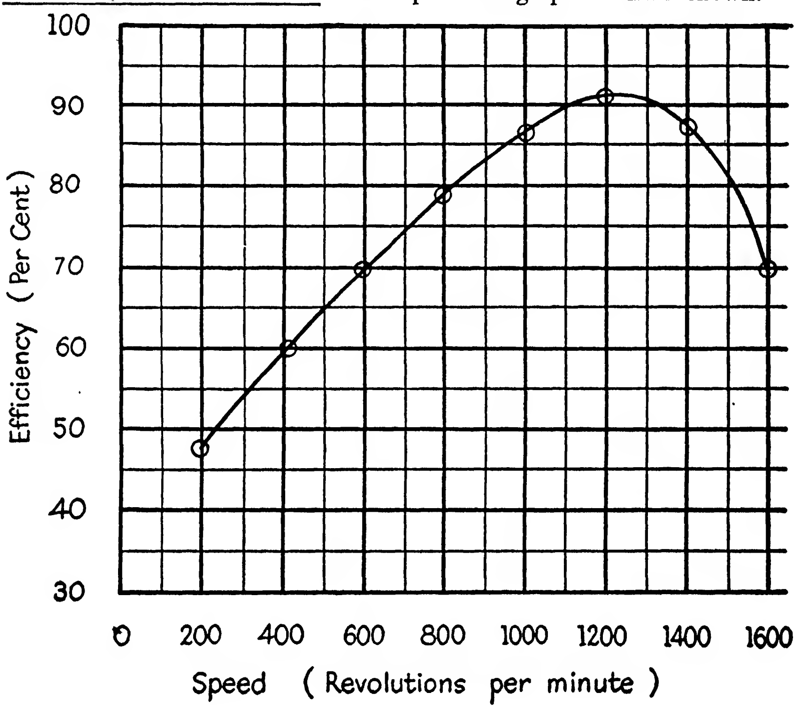
**What a Graph Can Do.** A graph, as the name itself suggests, can go a step further than the formula—it can make visible what the formula represents—it can give an actual *picture* of the mathematical relationship. The relationship literally becomes more graphic; the relative magnitudes of the variables become apparent to the eye, as do extreme maximum and minimum values, if any; so do the rates at which they change; trends become clear; extrapolation and interpolation become more meaningful; any special features of the relationship are emphasized; general types of relationships are recognizable; two or more relationships can frequently be directly compared with one another.

**Mathematical Functions.** In mathematics, a definite quantitative relation between two or more variables, whether expressed verbally, by a formula, or by a graph, is called a *functional relationship,* or simply a mathematical function. Each variable is said to be a function of the other. The word function, as used here, has nothing to do with use or purpose; it simply calls attention to the fact that the quantities in question are quantitatively related to each other in a definite manner. Each, in other words, depends upon the other (or others) for its numerical values. Thus if $V=32t^2$, then $V$ is a function of $t$, and $t$ is a function of $V$; or, the value of $V$ depends upon the particular value taken for $t$; and the value of $t$ depends upon the value of $V$.

It should be noted that in mathematics the word *curve* is used to designate any sort of a graph between two related variables, even though that graph may be a straight line, and not really "curved" at all. It also frequently happens that two variables, representing physical quantities or scientific measurements, may indeed be mutually dependent one upon the other, i.e., functionally related, and yet there is no simple or known formula to represent the relation between them. Such a relationship is called an *empirical function,* and its graph is called an empirical curve. Before discussing formula graphs (i.e., mathematical functions), we shall study empirical graphs a little further in order to understand mathematical graphs better.

**Empirical Curves.** Consider the accompanying table of values, which were determined by experimental measurements. It represents certain particular pairs of corresponding values of the two variables in question, which in this case are the speed of a certain machine (expressed in revolutions per minute) and the efficiency of the machine (expressed in per cents). Using the horizontal scale or axis for values of the speed (reading from left to right) and using the vertical axis for the efficiencies (reading from the bottom up), we then plot the graph as here shown.

| Speed | Efficiency (%) |
|---|---|
| 200 | 47.5 |
| 400 | 60.0 |
| 600 | 70.0 |
| 800 | 79.0 |
| 1000 | 86.0 |
| 1200 | 90.5 |
| 1400 | 87.5 |
| 1600 | 70.0 |




*Exercise 44.*

1. The solubility ($S$) of a certain chemical, as shown by the number of grams dissolved in a given amount of water, varied with the temperature ($T$) as shown in Table I. Plot the graph, using the horizontal scale for the values of the $S$. How many grams dissolved at 55°? At what temperature would 27 gm. be dissolved?

*Table I*

| $T$ | $S$ |
|---|---|
| 10 | 10 gm. |
| 20 | 12 |
| 30 | 16 |
| 40 | 21 |
| 50 | 30 |
| 60 | 50 |
| 65 | 65 |

*Table II*

| $T$ | $W$ |
|---|---|
| 5 | 6.8 |
| 10 | 9.4 |
| 15 | 12.8 |
| 20 | 17.2 |
| 25 | 22.9 |
| 30 | 29.2 |

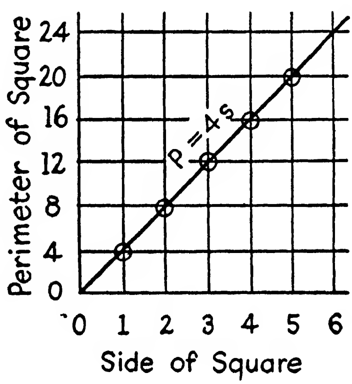
*Table III*

| $H$ | $P$ |
|---|---|
| 0 | 30.0 |
| 5,000 | 24.8 |
| 10,000 | 21.2 |
| 15,000 | 17.0 |
| 20,000 | 14.5 |
| 25,000 | 11.4 |
| 30,000 | 9.5 |
| 35,000 | 7.8 |

2. The weight ($W$ gm.) of water contained in a cubic meter of saturated air at various temperatures ($T$) is given in Table II. Plot the graph with the temperature along the horizontal scale. How many grams are contained in a cubic meter of air at 18°? at 28°? At what temperature does one cubic meter of saturated air contain 20 gms.? 25 gms.?
3. The atmospheric pressure ($P$) expressed in inches of mercury varies at different altitudes above sea level as shown in Table III. Plot the graph, using the vertical scale for the pressure. What is the pressure at the top of Mount Everest, 29,000 ft. high? How high is an airplane when its pressure gage indicates 22.5 inches?

**The Linear Function.** When two variables are so related that their graph is a straight line, the relation is said to be a *linear function*. Consider, for example, the relation between the side of a square and its perimeter. Whatever the length of its side, its perimeter is always 4 times as great; or, $P=4s$.

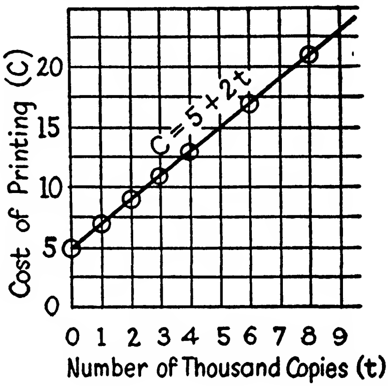
| $s$ | $P$ |
|---|---|
| 0 | 0 |
| 1 | 4 |
| 2 | 8 |
| 2½ | 10 |
| 3 | 12 |
| 4 | 16 |
| 5 | 20 |




When $s=0$, $P=0$; since, if there is no square, there can be no perimeter. Plotting these values on ruled paper, we obtain the accompanying linear graph. From this graph it is possible to "read off" many pairs of values of $s$ and $P$ that do not appear in the original table.

Or again, consider the cost of printing circulars, quoted at \$5 for setting the type, and \$2 per 1000 copies run off. The cost can be represented by the equation $C=5+2t$, where $t$=the number of thousand copies printed, and $C$=the total cost. The following set of values can readily be tabulated from the formula. When the graph of this table is

| $t$ | $C$ |
|---|---|
| 0 | 5 |
| 1 | 7 |
| 2 | 9 |
| 3 | 11 |
| 4 | 13 |
| 6 | 17 |
| 8 | 21 |




plotted, the curve is again seen to be a straight line. This time, however, it does not pass through the "0,0 point," but at the point where $t=0, C=5$. This, of course, mathematically represents the fact that even if no copies are printed the cost of setting type is \$5.

*Exercise 45.*

1. Plot the graph of $I=2.5C$, representing the relationship between inches ($I$) and centimeters. Use values of $C=0$, 2, 4, 6, 10. Use the horizontal axis for values of $C$.
2. Draw the graph of $E=IR$, considering a constant value of $R=20$; i.e., plot the graph of $E=20I$. Use values of $I=0$, 1, 2, 3, 4, 5, plotting them along the horizontal axis.
3. Plot the graph of $v=gt$, where $v$ is the velocity of a freely falling body in ft. per second, and $t$ is the time in seconds. Consider $g$ as constant and equal to 32 ft. per sec. Use values of $t=0$, 1, 2, 3, 5, 8, 10; use the horizontal axis for $t$.
4. Draw the graph of $D=RT$, where $R$ is constant and equal to 60 miles per hr., $T$ is the time in hours, and $D$ is the distance covered. Plot $T$ along the horizontal axis.
5. The circumference ($C$) of a circle is related to its diameter ($D$) as given by the formula $C=\pi D$, where $\pi=22/7$. Plot the graph, using values of $D=0$, 7, 14, 21, 28, 35, along the horizontal axis.
6. The amount due at simple interest on \$100 at 3% a year for various periods of time is given by the formula $A=100+3n$, where $n$ is the number of years. Plot the graph from $n=0$ to $n=6$; use the horizontal axis for $n$.

7. Plot the graph of $F=\frac{9}{5}C+32$, where $C$=the temperature expressed in Centrigrade degrees, and $F$=the temperature in Fahrenheit degrees. Use the horizontal axis for values of $C$.
8. Plot the graph of $V=v_0+at$, where $V$ is the final velocity in ft. per sec. and $t$ is the time in seconds, and where $v_0$, the initial velocity, is 600, and $a$, the acceleration, is 50 ft. per sec. Use values of $t=0, 2, 4, 6, 8, 10$ along the horizontal axis.

**Direct Variation.** Dependence or variation of this type, where the graph of the relation is a straight line, is called *direct variation*. The significance of the word "direct" is that as one variable increases, the other also increases in direct proportion, just as we saw in the section on ratio and proportion (Chap. I, Sect. 6). Whenever two variables *vary directly*, the relation may be expressed as

$$y=kx, \text{ or } \frac{y}{x}=k;$$

in this latter form it is seen that the *ratio* of the variables has a constant value, whatever their *individual* values may be. Recalling that the volume and temperature of a gas (at constant pressure) were found to be in *direct proportion;* thus

$$\frac{V_1}{T_1}=\frac{V_2}{T_2}, \text{ which can also be written as}$$

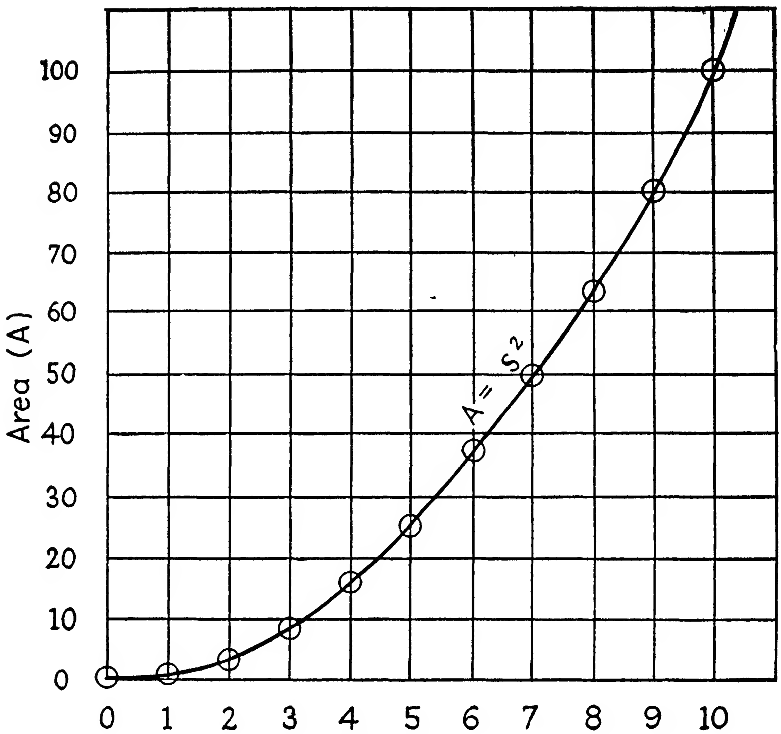
$$\frac{V_1}{T_1}=\frac{V_2}{T_2}=\frac{V_3}{T_3}=\frac{V_4}{T_4}=\ldots\ldots=k;$$

this means that the ratio of *any* volume to its *corresponding* temperature is the same ratio as *any other* volume is to *its* corresponding temperature; or, the *ratio being constant,* the relation is seen as an example of direct variation if it is simply rewritten as

$$\frac{V}{T}=k, \text{ or } V=kT.$$

**The Parabolic Function.** Some variables are related by a mathematical dependence which involves the square of one of them, as e.g., the relation between the area of a square and the length of its side, or $A=s^2$. Thus, from this formula we get the following table of values; then, from this tables of values we can plot the graph shown below. It is called a *parabolic* curve.

| $s$ | 0 | 1 | 2 | 3 | 4 | 5 | 8 | 10 |
|---|---|---|---|---|---|---|---|---|
| $A$ | 0 | 1 | 4 | 9 | 16 | 25 | 64 | 100 |

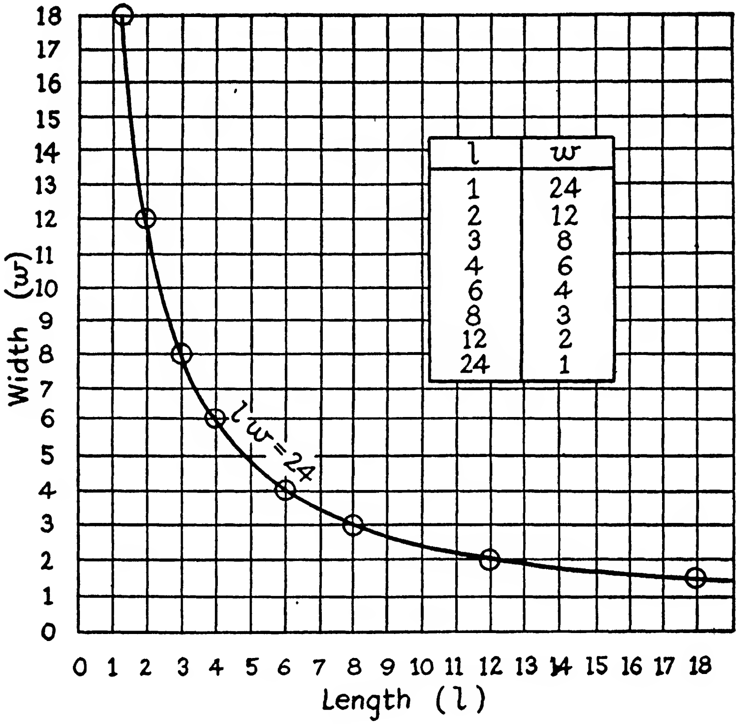



*Exercise 46.*

Plot the graph of each of the following formulas; in each case use the horizontal axis for the variable that is squared.

1. $S=at^2$, where $S$ is the distance ($S$) in feet moved by a body in $t$ seconds at a constant acceleration $a=10$ ft. per sec. Use values of $t=0, 1, 2, 3, 4$.
2. $A=\pi R^2$, where $A$ is the area of a circle, $R$=radius, and $\pi=22/7$. Use values of $R=0, 3\frac{1}{2}, 7, 10\frac{1}{2}, 14$.
3. $H=I^2R$, where $H$ is the heat produced by an electric current ($I$) that varies; consider the $R$ constant and equal to 10. Use values of $I=0, 1, 2, 3, 4, 5$.
4. The kinetic energy ($E$) of a moving automobile weighing 2,000 lb. is $E=\frac{1}{2}mv^2$, where $m=$ the weight of the car and $v$ its velocity in ft. per sec. Plot the graph for values of $v=0, 20, 40, 60, 80$.
5. The wind pressure ($P$) on a moving plane at 90° inclination, expressed in lb. per sq. ft., is given by the formula $P=.003\ v^2$, where $v$ is the wind velocity expressed in miles per hour. Plot the graph for values of $v=0, 10, 20, 30, 40, 50$.

**The Hyperbolic Function.** Another type of function frequently encountered is the *hyperbolic* relation, expressing inverse variation, such as the




relations $xy=k$, $A=lw$, or $R=\frac{D}{T}$. In each case, if one of the variables is held constant, the graph takes the characteristic form shown above. Consider first, the formula $A=lw$, where $l$ and $w$ are the length and width, respectively, of a rectangle having the area $A$. If we assume that the area $A$ is constant, equal say to 24 sq. in., then the graph of the formula $lw$ 24 is found as follows, since $l=\frac{24}{w}$, or $w=\frac{24}{l}$. This is the type of problem encountered by photoengravers in determining the possible dimensions for a rectangle of a given, constant area.

**Inverse Variation.** In this type of relation, neither variable can ever equal zero, although it can become as small as you please. The smaller either variable becomes, the larger is the value of the other, since their product is a constant; that is what is meant by *inverse variation.* It is exactly the same situation as was seen in the case of inverse proportion (Chap. I,

Sect. 6), where the pressure and volume of a gas at constant temperature were in *inverse* proportion, $\frac{P_1}{P_2}=\frac{V_2}{V_1}$ This can also be written as follows:

$$P_1\,V_1=P_2\,V_2=P_3\,V_3=P_4\,V_4=\;.\;.\;.\;=k;$$

i.e., *any* $P$ multiplied by its *corresponding* $V$ gives the *same constant product* as *any other* $P$ multiplied by *its* corresponding $V$. Hence the formula is $PV=k$, which is, as we now know, a hyperbolic curve.

*Exercise 47.*

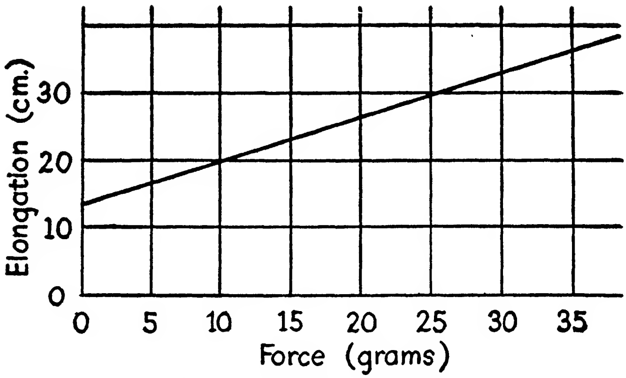
Plot the graph of each of the following formulas:

1. $xy=20$
2. $y=\frac{144}{x}$
3. $IR=120$
4. $PV=1000$
5. $T=\frac{100}{R}$
6. $500=Fs$

**Related Variables.** The dependence of varying quantities upon one another has already been discussed in connection with the formula. The value of either of two variables may properly be said to "depend" upon the other for its value. Quantities depending upon each other in this way are said to bear a functional relationship to each other; each is said to be a function of the other. Thus the volume occupied by a gas at constant temperature depends upon the pressure exerted; conversely, the pressure exerted depends upon the volume occupied by the gas. Hence $V$ is a function of $P$, or $P$ is a function of $V$. Similarly, the amount of expansion due to heat depends upon the temperature; the population of a community, upon the time; the fuel consumed in driving a locomotive, upon the speed; the quantity of a commodity sold, upon the price. In each instance the latter variable is conveniently regarded as the independent, and the former as the dependent, variable.

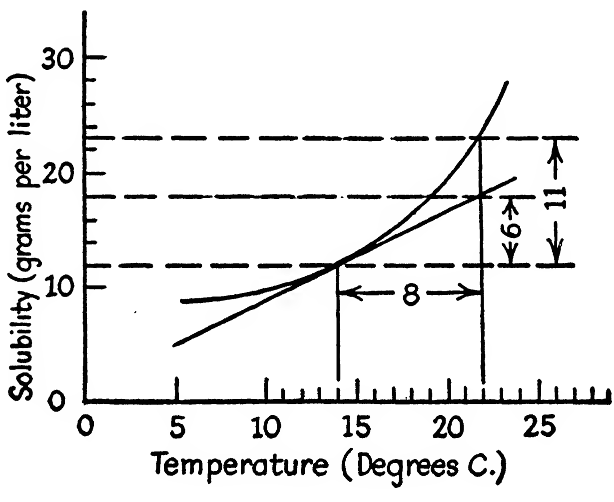
According to accepted convention, the independent variable is denoted by $x$ and its values are always measured along the horizontal scale, while the dependent variable is denoted by $y$, and its values are measured along the vertical scale. Hence, the independent variable may be regarded as freely assuming all values along the range of the horizontal axis, while the dependent variable necessarily varies in some definite way to correspond, as shown by the *changing height* of the graph.

**The Rate of Change Concept.** It is clear that for any given point on a graph, its horizontal distance from the vertical scale (abscissa) represents the magnitude of the independent variable, while the vertical distance above or below the horizontal scale (ordinate) represents the corresponding magnitude of the dependent variable. Thus the position of the curve with respect to the axes depicts the actual magnitudes of the variables. But in studying changing variables and functional relationships, it is frequently desirable to inquire as to the *rate* at which a quantity



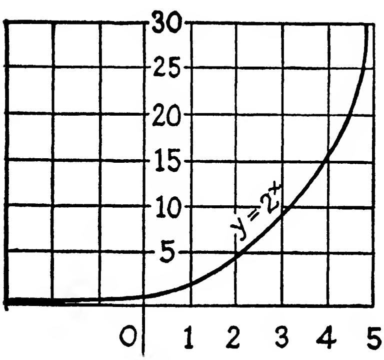

is changing, i.e., how fast it is increasing or decreasing, rather than how large or how small it is. Rate implies a ratio; a rate of change means the amount of change in the function (or dependent variable) per unit change in the independent variable. On the graph this means the amount of vertical rise or fall in the curve per horizontal unit, so that when discussing rates of change we are concerned with the *steepness* of the graph rather than with its actual height at any particular point. If the graph of a given function is a straight line, the function is obviously increasing at a constant rate, since the increase (or decrease) in the dependent variable is the same for every horizontal unit. Most quantities, however, change at varying instead of at constant rates, so that it becomes necessary to distinguish between an *average* rate and an *instantaneous* rate.

**Average Rate of Change.** An average rate during a definite interval does not necessarily imply that the rate during that interval need be uniform;



indeed, it may fluctuate considerably. An average rate during an interval merely means *the ratio of the net increase (or decrease) of the dependent variable during a given interval to the net change of the independent variable.* To find an average rate graphically, it is only necessary to read the amount of increase (or decrease) at the end of a given convenient interval and divide by the length of that interval. For example, the amount by which the solubility of a certain substance changed during the interval from 14° to 22° is seen to be 11 grams; hence the average rate is $^{11}/_{8}$ or 1.38 grams per degree. It is clear that the average rate of change will depend upon the length of the interval during which the average rate is observed, as well as where the interval is chosen.

**The Exponential Function.** One of the most interesting and important of all functions is the *exponential function,* $y=k^x$. It will be observed that it differs markedly from the ordinary *power function,* $y=x^k$; in the latter the exponent is merely a constant, while in the former it is the independent variable itself. Let us study carefully the graph of $y=2^x$. At first the curve rises very slowly, but gradually its rate of increase becomes greater and greater; as a matter of fact, the greater the value of $x$, the more rapid is the rate of growth of $y$. In this respect it resembles the graph of $A=P(1+i)^n$, which is the formula for the amount at compound interest, and is also of the type $y=k^x$. In both these curves, the *rate of increase of the dependent variable at any instant is proportional to the magnitude of the independent variable at the particular instant.* This is an extremely important characteristic of every exponential function. Otherwise expressed, the percentage of increase is constant throughout. Consequently any function of the type $y=ak^x$ is said to follow the *compound interest* law, or more picturesquely, the "snowball law," since it resembles the growth of a snowball rolling down a hill; it gathers more and more snow the farther down it rolls, and at any given instant is growing at a rate which is approximately proportional to the magnitude which it has already attained at that instant.



Graph of $y=2^x$

Exponential functions arise in connection with the speed of some chemical reactions, the rate of decomposition of radium, the reduction in speed of revolving wheels, the amount of belt friction on drums, the flow

of electric currents, the transmission of light under certain conditions, changes in atmospheric pressure with increase in elevation, and many other physical and technical phenomena.

## 11. LOGARITHMS

**Logarithms Are Exponents.** Consider the exponential equation $N=10^x$. How large is $N$? That depends upon the particular value selected for $x$; thus

$$\begin{array}{ll} \text{if } x=1, & N=10 \\ \text{if } x=2, & N=100 \\ \text{if } x=3, & N=1000 \\ \text{if } x=4, & N=10{,}000 \end{array}$$

*etc.*

What would be the value of $N$ if $x=1.5$? Clearly, since 1.5 is greater than 1, $N$ would be more than 10; and since 1.5 is less than 2, $N$ would be less than 100. In other words, when $x=1.5$, the value of $N$ must be *between* 10 and 100. For any value of $x$ equal to 1 plus some decimal, the value of $N$ must be between 10 and 100. Or again, if we know that the value of $N$ is some number between 100 and 1000, we know at once that $x$ must have a value somewhere between 2 and 3.

In the exponential form of the equation $N=10^x$, we refer to $N$ as the *number;* to 10, as the *base;* and to the exponent ($x$) as the *logarithm of the number N to the base 10*. This may be written as $x=\log_{10}N$. When expressed in this way, we speak of it as being in the "logarithmic form"; when written as $N=10^x$ we speak of it as the "exponential form." But the two equations are equivalent and interchangeable—two different ways of writing the same relationship.

Any number could be used as a base; thus

$$y=2^x;\ \text{or}\ \log_2 y=x$$
$$P=5^y;\ \text{or}\ \log_5 P=y$$
$$A=k^t;\ \text{or}\ \log_k A=t$$

For practical computational purposes, however, the base 10 is universally used, and for this reason the base is commonly omitted. Thus we write

$$\log 10{,}000=4,\ \text{instead of}\ \log_{10}10{,}000=4;$$
$$\text{or}\quad \log P=x,\ \text{instead of}\ \log_{10}P=x.$$

In other words, the logarithm of a given number is the exponent to which 10 must be raised to give that number. Logarithms are thus usually not whole numbers, but mixed numbers, and are invariably expressed in decimal form. Every number has a logarithm; also, every logarithm corresponds to some number.

**Characteristic and Mantissa.** From what has been said thus far, it is clear that very few logarithms are whole numbers—by far most of them are decimals whose approximate values are to be found in a table.

The part of the logarithm to the *left* of the decimal point, i.e., the integral part, is known as the *characteristic;* the decimal part is called the *mantissa.* Thus, the value of log 634=2.8021; the characteristic is "2" and the mantissa is ".8021."

Rule 1: If the number whose logarithm is being found is greater than 1, the characteristic is positive, and numerically *one less than the number of figures to the left of the decimal point* in the given number.

For example:

log 7.92 = .8987
log 79.2 =1.8987
log 792 =2.8987
log 79,200 =4.8987

Rule 2: If the number whose logarithm is to be found is less than 1, the characteristic is then negative and numerically *one greater than the number of zeros immediately following the decimal point;* but the mantissa is still positive. The two parts of the logarithm are not actually added together, but are written as shown below.

For example:

log .792 =—1.0+.8987=9.8987—10
log .0792 =—2.0+.8987=8.8987—10
log .00792=—3.0+.8987=7.8987—10

**Finding a Logarithm from the Table.** From these rules and illustrations it will be seen that the characteristic of a logarithm is determined solely by the position of the decimal point in the given number, and not by the particular sequence of digits; the mantissa, however, is quite independent of the position of the decimal point, depending instead upon the actual sequence of digits in the given number. Thus, for example, the mantissas of the respective logarithms of 264,500, 26.45, and .0002645 are identical; their logarithms differ only in the characteristics, and are, respectively, 5.4224, 1.4224, and 6.4224—10. In the "table of logarithms" only the mantissas are shown, the decimal point in front of each figure in the table being understood; the characteristic is supplied mentally, by inspection.

Example 1: Find from the table the value of log 467.

Solution: Look under the column headed "N," running down to "46"; then run across horizontally until reaching the column headed "7." The figure found is "6693." Hence log 467= 2.6693, since the 467 has 3 digits, and the required characteristic is 2.

Example 2: Find log .0803.

Solution: Look under "N" for 80; opposite 80, under "3," find "9047." Hence log .0803=8.9047—10.

# TABLE OF LOGARITHMS

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0000 | 0043 | 0086 | 0128 | 0170 | 0212 | 0253 | 0294 | 0334 | 0374 |
| 11 | 0414 | 0453 | 0492 | 0531 | 0569 | 0607 | 0645 | 0682 | 0719 | 0755 |
| 12 | 0792 | 0828 | 0864 | 0899 | 0934 | 0969 | 1004 | 1038 | 1072 | 1106 |
| 13 | 1139 | 1173 | 1206 | 1239 | 1271 | 1303 | 1335 | 1367 | 1399 | 1430 |
| 14 | 1461 | 1492 | 1523 | 1553 | 1584 | 1614 | 1644 | 1673 | 1703 | 1732 |
| 15 | 1761 | 1790 | 1818 | 1847 | 1875 | 1903 | 1931 | 1959 | 1987 | 2014 |
| 16 | 2041 | 2068 | 2095 | 2122 | 2148 | 2175 | 2201 | 2227 | 2253 | 2279 |
| 17 | 2304 | 2330 | 2355 | 2380 | 2405 | 2430 | 2455 | 2480 | 2504 | 2529 |
| 18 | 2553 | 2577 | 2601 | 2625 | 2648 | 2672 | 2695 | 2718 | 2742 | 2765 |
| 19 | 2788 | 2801 | 2833 | 2856 | 2878 | 2900 | 2923 | 2945 | 2967 | 2989 |
| 20 | 3010 | 3032 | 3054 | 3075 | 3096 | 3118 | 3139 | 3160 | 3181 | 3201 |
| 21 | 3222 | 3243 | 3263 | 3284 | 3304 | 3324 | 3345 | 3365 | 3385 | 3404 |
| 22 | 3424 | 3444 | 3464 | 3483 | 3502 | 3522 | 3541 | 3560 | 3579 | 3598 |
| 23 | 3617 | 3636 | 3655 | 3674 | 3692 | 3711 | 3729 | 3747 | 3766 | 3784 |
| 24 | 3802 | 3820 | 3838 | 3856 | 3874 | 3892 | 3909 | 3927 | 3945 | 3962 |
| 25 | 3979 | 3997 | 4014 | 4031 | 4048 | 4065 | 4082 | 4099 | 4116 | 4133 |
| 26 | 4150 | 4166 | 4183 | 4200 | 4216 | 4232 | 4249 | 4265 | 4281 | 4298 |
| 27 | 4314 | 4330 | 4346 | 4362 | 4378 | 4393 | 4409 | 4425 | 4440 | 4456 |
| 28 | 4472 | 4487 | 4502 | 4518 | 4533 | 4548 | 4564 | 4579 | 4594 | 4609 |
| 29 | 4624 | 4639 | 4654 | 4669 | 4683 | 4698 | 4713 | 4728 | 4742 | 4757 |
| 30 | 4771 | 4786 | 4800 | 4814 | 4829 | 4843 | 4857 | 4871 | 4886 | 4900 |
| 31 | 4914 | 4928 | 4942 | 4955 | 4969 | 4983 | 4997 | 5011 | 5024 | 5038 |
| 32 | 5051 | 5065 | 5079 | 5092 | 5105 | 5119 | 5132 | 5145 | 5159 | 5172 |
| 33 | 5185 | 5198 | 5211 | 5224 | 5237 | 5250 | 5263 | 5276 | 5289 | 5302 |
| 34 | 5315 | 5328 | 5340 | 5353 | 5366 | 5378 | 5391 | 5403 | 5416 | 5428 |
| 35 | 5441 | 5453 | 5465 | 5478 | 5490 | 5502 | 5514 | 5527 | 5539 | 5551 |
| 36 | 5563 | 5575 | 5587 | 5599 | 5611 | 5623 | 5635 | 5647 | 5658 | 5670 |
| 37 | 5682 | 5694 | 5705 | 5717 | 5729 | 5740 | 5752 | 5763 | 5775 | 5786 |
| 38 | 5798 | 5809 | 5821 | 5832 | 5843 | 5855 | 5866 | 5877 | 5888 | 5899 |
| 39 | 5911 | 5922 | 5933 | 5944 | 5955 | 5966 | 5977 | 5988 | 5999 | 6010 |
| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 40 | 6021 | 6031 | 6042 | 6053 | 6064 | 6075 | 6085 | 6096 | 6107 | 6117 |
| 41 | 6128 | 6138 | 6149 | 6160 | 6170 | 6180 | 6191 | 6201 | 6212 | 6222 |
| 42 | 6232 | 6243 | 6253 | 6263 | 6274 | 6284 | 6294 | 6304 | 6314 | 6325 |
| 43 | 6335 | 6345 | 6355 | 6365 | 6375 | 6385 | 6395 | 6405 | 6415 | 6425 |
| 44 | 6435 | 6444 | 6454 | 6464 | 6474 | 6484 | 6493 | 6503 | 6513 | 6522 |
| 45 | 6532 | 6542 | 6551 | 6561 | 6571 | 6580 | 6590 | 6599 | 6609 | 6618 |
| 46 | 6628 | 6637 | 6646 | 6656 | 6665 | 6675 | 6684 | 6693 | 6702 | 6712 |
| 47 | 6721 | 6730 | 6739 | 6749 | 6758 | 6767 | 6776 | 6785 | 6794 | 6803 |
| 48 | 6812 | 6821 | 6830 | 6839 | 6848 | 6857 | 6866 | 6875 | 6884 | 6893 |
| 49 | 6902 | 6911 | 6920 | 6928 | 6937 | 6946 | 6955 | 6964 | 6972 | 6981 |
| 50 | 6990 | 6998 | 7007 | 7016 | 7024 | 7033 | 7042 | 7050 | 7059 | 7067 |
| 51 | 7076 | 7084 | 7093 | 7101 | 7110 | 7118 | 7126 | 7135 | 7143 | 7152 |
| 52 | 7160 | 7168 | 7177 | 7185 | 7193 | 7202 | 7210 | 7218 | 7226 | 7235 |
| 53 | 7243 | 7251 | 7259 | 7267 | 7275 | 7284 | 7292 | 7300 | 7308 | 7316 |
| 54 | 7324 | 7332 | 7340 | 7348 | 7356 | 7364 | 7372 | 7380 | 7388 | 7396 |
| 55 | 7404 | 7412 | 7419 | 7427 | 7435 | 7443 | 7451 | 7459 | 7466 | 7474 |
| 56 | 7482 | 7490 | 7497 | 7505 | 7513 | 7520 | 7528 | 7536 | 7543 | 7551 |
| 57 | 7559 | 7566 | 7574 | 7582 | 7589 | 7597 | 7604 | 7612 | 7619 | 7627 |
| 58 | 7634 | 7642 | 7649 | 7657 | 7664 | 7672 | 7679 | 7686 | 7694 | 7701 |
| 59 | 7709 | 7716 | 7723 | 7731 | 7738 | 7745 | 7752 | 7760 | 7767 | 7774 |
| 60 | 7782 | 7789 | 7796 | 7803 | 7810 | 7818 | 7825 | 7832 | 7839 | 7846 |
| 61 | 7853 | 7860 | 7868 | 7875 | 7882 | 7889 | 7896 | 7903 | 7910 | 7917 |
| 62 | 7924 | 7931 | 7938 | 7945 | 7952 | 7959 | 7966 | 7973 | 7980 | 7987 |
| 63 | 7993 | 8000 | 8007 | 8014 | 8021 | 8028 | 8035 | 8041 | 8048 | 8055 |
| 64 | 8062 | 8069 | 8075 | 8082 | 8089 | 8096 | 8102 | 8109 | 8116 | 8122 |
| 65 | 8129 | 8136 | 8142 | 8149 | 8156 | 8162 | 8169 | 8176 | 8182 | 8189 |
| 66 | 8195 | 8202 | 8209 | 8215 | 8222 | 8228 | 8235 | 8241 | 8248 | 8254 |
| 67 | 8261 | 8267 | 8274 | 8280 | 8287 | 8293 | 8299 | 8306 | 8312 | 8319 |
| 68 | 8325 | 8331 | 8338 | 8344 | 8351 | 8357 | 8363 | 8370 | 8376 | 8382 |
| 69 | 8388 | 8395 | 8401 | 8407 | 8414 | 8420 | 8426 | 8432 | 8439 | 8445 |
| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

TABLE OF LOGARITHMS (*continued*)

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 70 | 8451 | 8457 | 8463 | 8470 | 8476 | 8482 | 8488 | 8494 | 8500 | 8506 |
| 71 | 8513 | 8519 | 8525 | 8531 | 8537 | 8543 | 8549 | 8555 | 8561 | 8567 |
| 72 | 8573 | 8579 | 8585 | 8591 | 8597 | 8603 | 8609 | 8615 | 8621 | 8627 |
| 73 | 8633 | 8639 | 8645 | 8651 | 8657 | 8663 | 8669 | 8675 | 8681 | 8686 |
| 74 | 8692 | 8698 | 8704 | 8710 | 8716 | 8722 | 8727 | 8733 | 8739 | 8745 |
| 75 | 8751 | 8756 | 8762 | 8768 | 8774 | 8779 | 8785 | 8791 | 8797 | 8802 |
| 76 | 8808 | 8814 | 8820 | 8825 | 8831 | 8837 | 8842 | 8848 | 8854 | 8859 |
| 77 | 8865 | 8871 | 8876 | 8882 | 8887 | 8893 | 8899 | 8904 | 8910 | 8915 |
| 78 | 8921 | 8927 | 8932 | 8938 | 8943 | 8949 | 8954 | 8960 | 8965 | 8971 |
| 79 | 8976 | 8982 | 8987 | 8993 | 8998 | 9004 | 9009 | 9015 | 9020 | 9025 |
| 80 | 9031 | 9036 | 9042 | 9047 | 9053 | 9058 | 9063 | 9069 | 9074 | 9079 |
| 81 | 9085 | 9090 | 9096 | 9101 | 9106 | 9112 | 9117 | 9122 | 9128 | 9133 |
| 82 | 9138 | 9143 | 9149 | 9154 | 9159 | 9165 | 9170 | 9175 | 9180 | 9186 |
| 83 | 9191 | 9196 | 9201 | 9206 | 9212 | 9217 | 9222 | 9227 | 9232 | 9238 |
| 84 | 9243 | 9248 | 9253 | 9258 | 9263 | 9269 | 9274 | 9279 | 9284 | 9289 |
| 85 | 9294 | 9299 | 9304 | 9309 | 9315 | 9320 | 9325 | 9330 | 9335 | 9340 |
| 86 | 9345 | 9350 | 9355 | 9360 | 9365 | 9370 | 9375 | 9380 | 9385 | 9390 |
| 87 | 9395 | 9400 | 9405 | 9410 | 9415 | 9420 | 9425 | 9430 | 9435 | 9440 |
| 88 | 9445 | 9450 | 9455 | 9460 | 9465 | 9469 | 9474 | 9479 | 9484 | 9489 |
| 89 | 9494 | 9499 | 9504 | 9509 | 9513 | 9518 | 9523 | 9528 | 9533 | 9538 |
| 90 | 9542 | 9547 | 9552 | 9557 | 9562 | 9566 | 9571 | 9576 | 9581 | 9586 |
| 91 | 9590 | 9595 | 9600 | 9605 | 9609 | 9614 | 9619 | 9624 | 9628 | 9633 |
| 92 | 9638 | 9643 | 9647 | 9652 | 9657 | 9661 | 9666 | 9671 | 9675 | 9680 |
| 93 | 9685 | 9689 | 9694 | 9699 | 9703 | 9708 | 9713 | 9717 | 9722 | 9727 |
| 94 | 9731 | 9736 | 9741 | 9745 | 9750 | 9754 | 9759 | 9763 | 9768 | 9773 |
| 95 | 9777 | 9782 | 9786 | 9791 | 9795 | 9800 | 9805 | 9809 | 9814 | 9818 |
| 96 | 9823 | 9827 | 9832 | 9836 | 9841 | 9845 | 9850 | 9854 | 9859 | 9863 |
| 97 | 9868 | 9872 | 9877 | 9881 | 9886 | 9890 | 9894 | 9899 | 9903 | 9908 |
| 98 | 9912 | 9917 | 9921 | 9926 | 9930 | 9934 | 9939 | 9943 | 9948 | 9952 |
| 99 | 9956 | 9961 | 9965 | 9969 | 9974 | 9978 | 9983 | 9987 | 9991 | 9996 |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

**Interpolation.** If the number whose logarithm is sought should contain more than three significant figures, the logarithm is found by a method known as *interpolation*. This is based upon the idea of direct proportion; i.e., it is assumed that the logarithm of a number "halfway between" two numbers is halfway between their logarithms; etc. Although such direct variation between numbers and their logarithms is not strictly true, the approximation is accurate enough for ordinary purposes of calculation.

EXAMPLE 1: Find log 2834.

SOLUTION: log 2830=3.4518
log 2840=3.4533
Interval between numbers="10"
Difference between logarithms=.0015
Given number $N$=2834 is .4 of the interval 10;
.4×.0015=.0006.
Hence, log 2834=3.4518+.0006=3.4524, *Ans.*

EXAMPLE 2: Find log .06178.

SOLUTION: log .06170=8.7903—10
log .06180=8.7910—10
Difference between logs=.0007
Given $N$=.06178 is .8 of "interval" between .06170 and .06180; thus .8×.0007=.00056, or .0006, is added to the mantissa .7903, giving .7909.
Hence log .06178=8.7909—10, *Ans.*

EXAMPLE 3: Find log 12.476

SOLUTION: log 12.4=1.0934
log 12.5=1.0969
Difference between logs=.0035
.76×.0035=.0027.
Hence log 12.476=1.0961, *Ans.*

*Exercise 48.*

Find the logarithms of the following numbers:

| | | |
|---|---|---|
| 1. 932 | 5. .000526 | 9. .6932 |
| 2. 408.5 | 6. 643,000 | 10. .00045 |
| 3. .504 | 7. 8.037 | 11. 4,806 |
| 4. 71.68 | 8. .00341 | 12. 236.96 |

**Finding the Number When Its Logarithm Is Given.** If the logarithm of a number is given and we wish to determine the number itself, this reverse process is called finding the *antilogarithm*. The principle of interpolation is used, but in the reverse way; the procedure follows:

(1) In the table, find the two mantissas between which the given mantissa lies, and write the three figures corresponding to the smaller of these two mantissas.

(2) Next find the difference between these two consecutive mantissas, and also the difference between the smaller of them and the given mantissa whose antilog we are seeking.
(3) Divide the latter difference by the former, carrying the division to the nearest tenth.
(4) Annex the figure so found to the three figures already found, making it the fourth figure of the antilogarithm.
(5) Locate the decimal point in the antilogarithm by inspection of the given logarithm.

It sounds far more difficult than it really is.

EXAMPLE 1: Find the antilog of 2.3942.

SOLUTION: The given mantissa .3942 lies between 3927 and 3945; thus the first three figures of the antilog are "247."

$$\begin{array}{r} .3945 \\ \underline{.3927} \\ .0018 \end{array} \qquad \begin{array}{r} .3942 \\ \underline{.3927} \\ .0015 \end{array} \qquad \frac{.0015}{.0018}=.8$$

hence antilog=247.8, *Ans.*

EXAMPLE 2: Find the antilog of 7.5029—10.

SOLUTION: Antilog lies between 318 and 319.

$$\begin{array}{r} .5038 \\ \underline{.5024} \\ .0014 \end{array} \qquad \begin{array}{r} .5029 \\ \underline{.5024} \\ .0005 \end{array} \qquad \frac{.0005}{.0014}=.4$$

hence antilog=.003184, *Ans.*

### *Exercise 49.*

Find the number whose logarithm is:

| | | |
|---|---|---|
| 1. 1.3806 | 5. 1.3989 | 9. 2.4416 |
| 2. .6432 | 6. 9.3001—10 | 10. 4.7215 |
| 3. 2.4237 | 7. .7625 | 11. 7.9103—10 |
| 4. 9.6145—10 | 8. 8.4182—10 | 12. 3.0462 |

**Laws of Exponents Applied to Logarithms.** By applying the laws of exponents to logarithmic expressions, we find that if $M$ and $N$ are any two numbers, then:

*Law I.* $\log (MN)=\log M+\log N$.

Suppose we let $\log M=x$,
and $\log N=y$.
then $10^x=M$, and $10^y=N$.
But $(MN)=(10^x)\cdot(10^y)=10^{(x+y)}$;
hence $\log (MN)=x+y$,
or $\log (MN)=\log M+\log N$.

In other words, *the logarithm of a product equals the sum of the logarithms of its factors.*

For example: $\log 21 = \log 7 + \log 3$
$\log 5000 = \log 1000 + \log 5$
$\log (372 \times 28.1) = \log 372 + \log 28.1$
$\log Prt = \log P + \log r + \log t$

NOTE: This property of logarithms, viz.
$\log (PQ) = \log P + \log Q,$
should not be confused with the following:
$\log P + \log Q \neq \log (P+Q).$

In short, the *sum of two logarithms does not* equal the logarithm of their sum; neither does the logarithm of the *sum of two numbers* equal the sum of their logarithms.

*Law II.* $\log\left(\frac{M}{N}\right) = \log M - \log N$

That is, *the logarithm of a quotient equals the logarithm of the numerator diminished by the logarithm of the denominator.*

NOTE: This should not be misinterpreted; $\log (M-N) \neq \log M - \log N$, and $\frac{\log M}{\log N} \neq \log M - \log N$. In short, the logarithm of the difference of two numbers is the logarithm of *their* difference, not the difference of their logarithms. Again, the ratio of two logarithms is simply a number; it is not the same as the logarithm of a ratio, and is therefore not equal to the difference of two logarithms.

*Law III.* $\log (N)^p = p(\log N)$

*Law IV.* $\log(N)^{\frac{1}{p}} = \log\sqrt[p]{N} = \frac{1}{p}\ (\log N)$

Thus, for example, we have:

$\log r^2 = 2 \log r$
$\log v^3 = 3 \log v$
$\log \sqrt{d} = \frac{1}{2} \log d$
$\log \sqrt[3]{k} = \frac{1}{3} \log k$
$\log p^{3/2} = \frac{3 \log p}{2}$
$\log \sqrt{\pi r^2} = \frac{1}{2} \log \pi + \log r$

**Multiplying and Dividing by Using Logarithms.** The chief practical value of logarithms is in shortening the multiplication and division of numbers. When skill in their use has been achieved, the time and labor required by longhand multiplication and division in extensive computations is very materially lessened, as the following examples will show.

EXAMPLE 1: Multiply $396.2 \times 8.735 \times .07434$.

SOLUTION:

$$\begin{array}{rl}
\log 396.2 = & 2.5979 \\
\log 8.735 = & .9413 \\
\log .07432 = & \underline{8.8712-10} \\
& 12.4104-10
\end{array}$$

antilog $2.4104=257.3$, *Ans.*

EXAMPLE 2: Find the value of

$$\frac{.376\times 4.018\times 57.64}{.0675\times 3.1416}.$$

SOLUTION:

$$\begin{array}{rl}
\log .376 = & 9.5752-10 \\
\log 4.018 = & .6040 \\
\log 57.64 = & \underline{1.7607} \\
& 11.9399-10 \\
& \underline{9.3264-10} \\
& 2.6135
\end{array}
\qquad
\begin{array}{rl}
\log .0675 = & 8.8293-10 \\
\log 3.1416 = & \underline{.4971} \\
& 9.3264-10
\end{array}$$

antilog $2.6135=410.8$, *Ans.*

*Exercise 50.*

Find, by using logarithms, the value of:

1. $\dfrac{622.5\times .0317}{14.8}$
2. $\dfrac{.00905}{2.54\times 3614}$
3. $\dfrac{46.22\times 309.4}{162\times .084}$
4. $2\times \frac{22}{7}\times 18.6\times 35.8$
5. $\frac{1}{365}\times .0525\times 1650$
6. $\dfrac{26.8\times .00048\times 127.5}{709.2\times .486}$

**Using Logarithms to Find Roots and Powers.** Another extremely practical use of logarithms is in finding roots and powers of numbers.

EXAMPLE 1: Find $\sqrt[5]{62.4}$.

SOLUTION: Let $x=(62.4)^{1/5}$

$\log x=\frac{1}{5}(\log 62.4)$

$\log x=(\frac{1}{5})(1.7952)=.3590$

$x=2.286$, *Ans.*

EXAMPLE 2: Compute the value of $(1.055)^{12}$.

SOLUTION: Let $x=(1.055)^{12}$

$\log x=12(\log 1.055)$

$\log x=(12)(.0232)=.2784$

$x=1.898$, *Ans.*

EXAMPLE 3: What is the value of $x=\dfrac{.0046\times\sqrt[3]{85.7}}{(.063)^2}$.

SOLUTION:

$$\begin{aligned}
\log .0046 &= 7.6628-10\\
\tfrac{1}{3}\log 85.7 &= \underline{\phantom{0}.6443}\\
&\phantom{=} 8.3071-10\\
2\log .063 &= \underline{17.5986-20}\\
\log x &= \phantom{0}.7085\\
x &= 5.111, \textit{Ans.}
\end{aligned}$$

EXAMPLE 4: In the formula $t=\sqrt{\frac{2s}{g}}$, find $t$ when $s=1155$ and $g=32.2$.

SOLUTION: $t=\left(\frac{2s}{g}\right)^{\frac{1}{2}}$

$$\begin{aligned}
\log t &= \tfrac{1}{2}\,[\log 2+\log s-\log g]\\
&= \tfrac{1}{2}\,[\log 2+\log 1155-\log 32.2]\\
\log 2 &= .3010\\
\log 1155 &= \underline{3.0626}\\
&\phantom{=} 3.3636\\
\log 32.2 &= \underline{1.5079}\\
&\phantom{=} 2\overline{)1.8557}\\
\log t &= .9279\\
t &= 8.47, \textit{Ans.}
\end{aligned}$$

*Exercise 51.*

Using logarithms, compute each of the following:

1. $\sqrt[3]{756}$
2. $\sqrt[5]{895}$
3. $(.562)^5$
4. $\sqrt[3]{9.47}$
5. $(6.29)^{\frac{2}{3}}$
6. $\sqrt[4]{24.38}$
7. $(1.045)^9$
8. $(40.21)^3$
9. $\sqrt{.000394}$
10. $(8.9)^3\times(27.84)$
11. $\sqrt[3]{162.4}\times(85.36)$
12. $(.364)^2\times 29.25\times\sqrt[5]{162.4}$
13. $\sqrt[3]{\dfrac{215.2}{3.14\times 4.57}}$
14. The formula for the volume of a cylinder is $V=\pi R^2H$; find $R$ when $V=906.0$, $H=14.6$ and $\pi=3.142$.
15. The volume of a sphere is given by $V=\frac{4}{3}\pi R^3$; find to the nearest hundredth the diameter of a sphere whose volume is 85.4 cu. in., using $\pi=3.142$.

**Solving Exponential Equations by Use of Logarithms.** Still another useful application of logarithms is in connection with the solution of *exponential equations,* i.e., an equation in which the variable (or unknown quantity) is an exponent. Such equations frequently arise in engineering and technical problems.

Example 1: Solve for $x$: $18=(4.5)^x$.

Solution: $\log 18=(x)(\log 4.5)$

$$x=\frac{\log 18}{\log 4.5}=\frac{1.2553}{.6532}=1.92, \textit{ Ans.}$$

Example 2: Solve: $5^k=10^{k+1}$.

Solution: $k(\log 5)=(k+1)(\log 10)$

$$\log 5=.6990$$
$$\log 10=1.0000$$
$$.699k=k+1$$
$$k=-\frac{1}{.301}=-3.33, \textit{ Ans.}$$

*Exercise 52.*

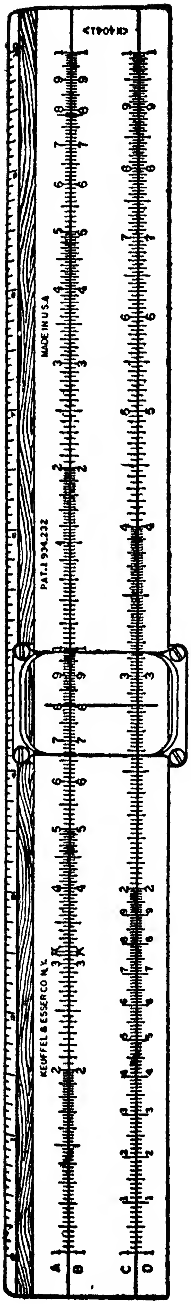
Solve the following exponential equations:

1. $5^x=18$
2. $1.5=(1.03)^x$
3. $12.8=75^{x_2}$
4. $(3.02)^x=100$
5. $(3.4)^x=22.84$
6. $(2^x)(10^x)=30$
7. $\sqrt[x]{12}=4$
8. $(1.04)^{t+1}=6.8$
9. A principal of \$400 at compound interest for a certain length of time at 3½% a year, compounded annually, amounted to \$750. Find the time.
10. A machine originally costing \$1800 depreciates in value each year 15% of its value at the beginning of the year. What is its value at the end of 6 years? [*Hint:* $V=1800\ (.85)^6$]. In how many years will it be worth \$900?

## 12. THE SLIDE RULE

**Mechanical Computation.** Historically speaking, man has made use of various sorts of mechanical aids to computation throughout the ages. The slide rule is one such mechanical device. It was invented some 300 years ago by an English mathematician, William Oughtred; subsequently improved, in its modern form it was invented about 1850 by Amédée Mannheim, and is now known by his name. In addition to the simple Mannheim rule, there are available today some ten or twelve other types, more elaborate and more complicated, and designed for a variety of mathematical computations as well as for special purposes. In the present discussion we shall confine our attention to the simple Mannheim rule.

**Description of the Rule.** Generally made of wood, xylonite or celluloid, the slide rule is from 5 to 10 inches in length. It consists of three parts: the rule, which is grooved; the slide, which is carefully fitted so that it slides easily in the grooved rule from left to right and vice versa; and the hairline runner. The slide and the rule are both faced with accurately graduated scales.



In all, there are four rows of figures on the front of the slide rule:

*Row A:* two complete logarithmic scales, on the rule, giving directly squares and square roots.

*Row B:* two complete logarithmic scales, exactly like A, but on the slide.

*Row C:* a single logarithmic scale, also on the slide.

*Row D:* another single logarithmic scale, exactly like C, but on the rule.

On the back of the slide there are to be found three more scales:

*Row S:* a trigonometric scale of sines, used in conjunction with the A and B scales.

*Row T:* a trigonometric scale of tangents, used in conjunction with the C and D scales.

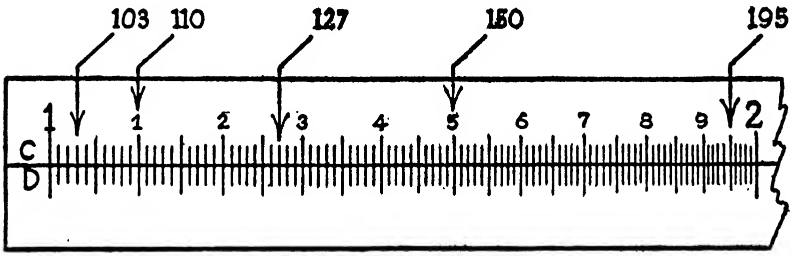
*Row L:* a scale of equal parts, used to find the common logarithms of numbers.

**The C and D Scales.** Confining ourselves to the front of the slide rule, and noting the two scales marked C and D, respectively, we find that the graduations in each case begin with 1 at the extreme left and are numbered 1, 2, 3, 4, etc., from the left to 8, 9, 1 at the right. The reason that the figure at the extreme right is 1 and not 10 is that *all* the figures on the scales, as marked, are *arbitrary.* Thus the initial 1 at the left index may represent 10, 100, 0.1, 0.01, etc., depending upon the particular computation, but once the value of the left index has been chosen, the same ratio must be observed throughout; e.g., if we start reading it as 10, then the other figures are to be read 20, 30, 40, etc.; if we commence with .01, then the other figures are read as .02, .03, .04, etc. It will also be observed that the *actual* space or distance from 1 to 2 is the same as that from 2 to 4, and also as that from 4 to 8; that is what makes it a *logarithmic* scale.

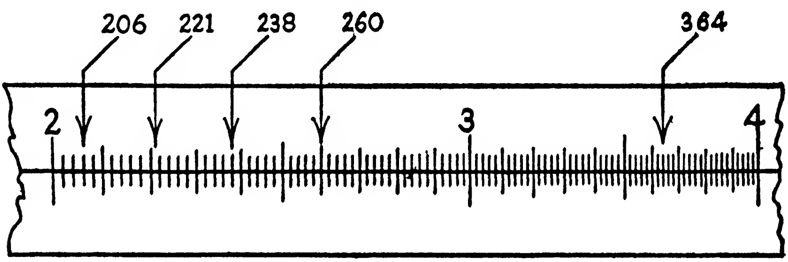
The segment between 1 and 2 is subdivided into 10 parts, each succeeding part decreasing slightly in size; each of these 10 subdivisions is again divided into 10 smaller divisions. If space permitted, these

subdivisions would be carried out all along, but, because of their decreasing size, they are later subdivided only into halves or fifths, which must be interpreted as decimal divisions and not as fractional ones. Nor are they all marked with numerals, since this would tend to overcrowd the rule and make it difficult to read; moreover, as one acquires skill and facility in using the slide rule, these additional numbers are not really needed at all.

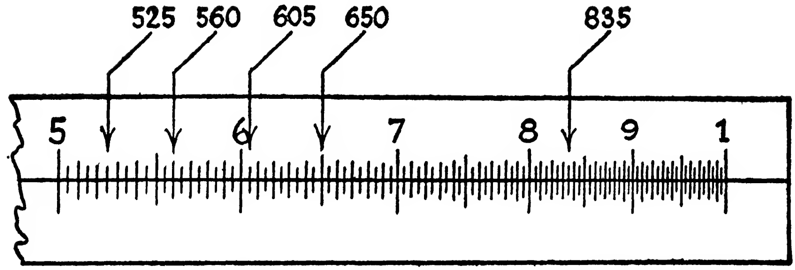
To learn to read these scales, study the following sketches given of portions of the rule and slide, noting especially the "sample readings" indicated.




It will be seen that this portion of the C and D scales (between 1 and 2) indicates accurately numbers which contain three digits, if the first digit is 1. If a number contains four digits (the last one not being zero), the number can be read accurately to three digits, but the fourth digit must be estimated. Do not be concerned about this; *always estimate to the maximum degree the smallest subdivision when reading a slide rule.*




Turning our attention next to the portion of the C and D scales between 2 and 4, it will be shown how these figures are read. Any number beginning with 2 and 3 is read from this part of the scale. In this portion of the scale each large unnumbered division represents 1/10 of 100, or 10; furthermore, since there are five unnumbered divisions between any two large unnumbered divisions, each small unnumbered division represents 2. Thus all even numbers between 200 and 400 can be read accurately, but all odd numbers must be estimated.




Finally, the portion of the C and D scale from 4 to 10 is used to locate any number whose first digit is 4, 5, 6, 7, 8 or 9. The first two digits can always be read accurately; the third digit can be read accurately only if it is a 5, otherwise it must be estimated.

**The Principle of Proportion.** The slide rule is not designed for addition and subtraction; its unique adaptability is for multiplication, division, proportions, powers and roots. It is a basic principle of the slide rule that, no matter where the slide is placed, *all* the numbers on the slide bear the same ratio to their corresponding or coinciding numbers on the rule. This holds true of the C and D scales, and also of the A and B scales. For example, pull the slide out to the right until 1 on C corresponds exactly with, or is "over," 2 on D; you will then note that the ratio 1:2 exists between every pair of coinciding numbers on the C and D scales, respectively. Thus if we set the first two terms of a proportion against each other on the slide and the rule, we find the third term on the slide coinciding with the fourth term on the rule. Hence:

$$\begin{pmatrix}\text{any number}\\ \text{on C}\end{pmatrix} : \begin{pmatrix}\text{the number}\\ \text{under it on D}\end{pmatrix} = \begin{pmatrix}\text{any other}\\ \text{number on C}\end{pmatrix} : \begin{pmatrix}\text{the number}\\ \text{under it on D}\end{pmatrix}$$

This principle of proportion can also be expressed as follows, giving a rule of procedure for using the slide rule to solve for the fourth term of a proportion:

| | | |
|---|---|---|
| C | set first term | under third term |
| D | over second term | find fourth term |

EXAMPLE: Solve for $x$: $\dfrac{3}{8\frac{1}{2}} = \dfrac{15}{x}$.

SOLUTION:

| | | | |
|---|---|---|---|
| C | set 3 | under 15 | *Ans.*, 42.5 |
| D | over 8.5 | find 42.5 | |

**Multiplication.** Since the multiplication of two factors, say $4 \times 7$, is the same as finding $x$ in the proportion $1:4=7:x$, the rule for multiplication on the slide rule is simply this:

| | | |
|---|---|---|
| C | set 1 | under the other factor |
| D | over one factor | find their product |

**Example** 1: Multiply $14 \times 6\frac{1}{2}$.

Solution:

| | | | |
|---|---|---|---|
| C | set 1 | under 65 | *Ans.*, 91 |
| D | over 14 | find 91 | |

Note: The location of the decimal point in the answer is most readily achieved by inspection of the original numbers rather than by any other rule; thus it is obvious that $14 \times 6\frac{1}{2}$ is approximately equal to 100, so that when reading the value "9—1" it would be read as 91 rather than as 9.1 or 910.

**Example** 2: Find the product of $5\frac{1}{4} \times 24$.

Solution:

| | | | |
|---|---|---|---|
| C | set 1 | under 24 | *Ans.*, 126 |
| D | over 525 | find 126 | |

Note: If when drawing the slide out *to the right* in order to set 1 on C over one of the factors on D the other factor on C extends beyond the right terminal 1 on D, then the slide is drawn to the left and the right terminal 1 on C is set over the first factor on D.

**Example** 3: Multiply: $17.6 \times .0428$.

Solution:

| | | | |
|---|---|---|---|
| C | set 1 | under 428 | *Ans.*, 0.753 |
| D | over 176 | 753 | |

Note 1: Since $18 \times .04 = .72$, the decimal point is placed before the 7 in reading "753."

Note 2: In setting 428, the terminal "8" is *estimated* as nearly as possible between "425" and "430"; likewise, when reading the product "753" on the D scale, the terminal "3" is also estimated as closely as possible. In this case, the product might have been read as .752 or .754 instead of .753, which would be an error of only one part in a thousand; the actual product is .75328. But since the factors are only given to three significant figures each, the product is adequately expressed as .753.

**Division.** The procedure for dividing one number by another is quite similar, as might be expected, since division is the reverse of multiplication. Thus the rule is:

| | | |
|---|---|---|
| C | set divisor | under 1 |
| D | over dividend | find quotient |

EXAMPLE 1: Divide 64 by 4½.

SOLUTION:

| | | | |
|---|---|---|---|
| C | set 45 | under 1 | *Ans.*, 14.25 |
| D | over 64 | find 1425 | |

EXAMPLE 2: Find $0.256 \div 33.2$.

SOLUTION:

| | | | |
|---|---|---|---|
| C | set 332 | under 1 | *Ans.*, 0.00771 |
| D | over 256 | find 771 | |

EXAMPLE 3: Divide 80.3 by .049.

SOLUTION:

| | | | |
|---|---|---|---|
| C | set 49 | under 1 | *Ans.*, 1639 |
| D | over 803 | find 1639 | |

**Combined Multiplication and Division.** Here the extreme utility of the slide rule becomes even more evident, as the following illustrations will make abundantly clear.

EXAMPLE 1: Find: $18 \times 4.2 \times .17$.

SOLUTION:

| | | | | |
|---|---|---|---|---|
| C | set 1 | bring runner to 42 | bring 1 to runner | under 17 |
| D | on 18 | | | find 1285 |

*Ans.*, 12.85

EXAMPLE 2: Find: $\dfrac{27.2 \times 44 \times .88}{.73 \times 6.4}$

SOLUTION:

| | | | | |
|---|---|---|---|---|
| C | set 73 | runner to 44 | 64 to runner | under 88 |
| D | on 272 | | | find 2255 |

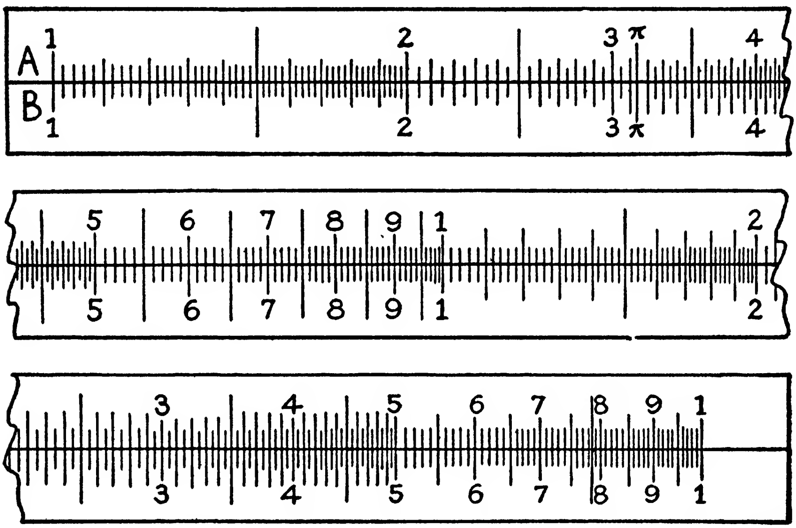
*Ans.*, 225.5

*Exercise 53.*

Using the slide rule, find the following:

1. $39.6 \times 24$
2. $87 \times 6.5$
3. $6.21 \times 9.24$
4. $7.4 \div 32$
5. $60.5 \div 0.49$
6. $4.62 \times .866 \times 29$
7. $3.08 \times 144 \times 1.6$
8. $28.5 \times 3.14 \times 0.045$
9. $835 \div 21.8$
10. $.32 \div 0.077$
11. $\dfrac{63 \times 22}{17.5}$
12. $\dfrac{8.02 \times 30.6}{68 \times 25}$
13. $\dfrac{46.5 \times 3.14}{80.2 \times .35}$
14. $\dfrac{.707 \times 18.4}{2.54}$

**Squares and Square Roots.** Upon examining the upper scales A and B, it will be found that all the numbers on both the A and B scales are the *squares* of their coinciding numbers on the C and D scales; conversely, all the numbers on the C and D scales are the *square roots* of their coinciding numbers on the A and B scales. Thus, if the runner is set to 3 on the D scale, the hairline coincides with 9 on the A scale; this may




be read as $3^2=9$, or $\sqrt{9}=3$; also as $30^2=900$, or $\sqrt{900}=30$; also as $300^2=90000$, or $\sqrt{90000}=300$; etc. Furthermore, if the runner is set to 5 on the D scale, the hairline will coincide with 25 on the A scale (not 2.5); i.e., $5^2=25$, or $\sqrt{25}=5$; also, $50^2=2500$, or $\sqrt{2500}=50$; etc. It should further be observed that the A scale is not graduated in the same manner as the D scale, since the A scale consists of *two parts,* the left- and right-hand scales, each running from "1" to "1." But if the left index of the left half of the A scale is "1," then the right terminal index of that half of the scale is "10," and the "2" in the right half of the A scale is then "20," etc.

In other words, to find the square root of a number, we locate that number on the A scale, observing the following rules; then set the runner on the number (on the A scale) whose root is to be found, and under it, on the D scale, read the desired square root:

*Rule 1:* To find the square root of a number having an *even* number of digits to the left of the decimal point, use the *right half* of the A scale; e.g., to find $\sqrt{62}$, $\sqrt{38.4}$, $\sqrt{6531.5}$, etc.

*Rule 2:* To find the square root of a number having an *odd* number of digits to the left of the decimal point, use the *left half* of the A scale; e.g., to find $\sqrt{6}$, $\sqrt{182}$, $\sqrt{3.1416}$, $\sqrt{856.2}$, etc.

*Rule 3:* To find the square root of a decimal having an *even* number of zeros to the right of the decimal point, use the *right half* of the A scale; e.g., $\sqrt{0.35}$, $\sqrt{0.0042}$, $\sqrt{0.831}$, $\sqrt{0.0000562}$, etc.

*Rule 4:* To find the square root of a decimal having an *odd* number of zeros to the right of the decimal point, use the *left half* of the A scale; e.g., $\sqrt{0.0269}$, $\sqrt{0.00048}$, $\sqrt{0.086}$, etc.

Example 1: Find the square root of 39.5.

Solution: Since 39 contains an even number of digits to the left of the decimal point, we use the *right half* of the A scale. Set runner on 395 on right side of A scale; under hairline on D scale read "628."

*Ans.,* $\sqrt{39.5}=6.28$.

Example 2: Find $\sqrt{431}$.

Solution: Since 431 contains an odd number of digits to the left of the decimal point, we use the *left half* of the A scale. Set runner on 431 on left side of A scale; under hairline on D scale read "2075."

*Ans.,* $\sqrt{431}=20.75$.

Example 3: Find $\sqrt{0.88}$.

Solution: Since the decimal has no zeros, i.e., an even number of zeros, to the right of the decimal point, we use the *right half* of the A scale. Set runner on 88 on right side of A scale; under hairline read "938."

*Ans.,* $\sqrt{0.88}=0.938$.

Example 4: Find $\sqrt{0.0825}$.

Solution: Since the decimal has an odd number of zeros to the right of the decimal point, we use the *left half* of the A scale. Set runner on 825 on left side of A scale; under hairline read "287."

*Ans.,* $\sqrt{0.0825}=0.287$.

*Exercise 54.*

Find the square root of each of the following by using the slide rule:

| | | |
|---|---|---|
| 1. 44 | 5. 8.25 | 9. 3450 |
| 2. 24.8 | 6. .846 | 10. 0.005 |
| 3. 70.5 | 7. 0.00032 | 11. 3.07 |
| 4. 137 | 8. 0.0643 | 12. 0.0076 |

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# CHAPTER III

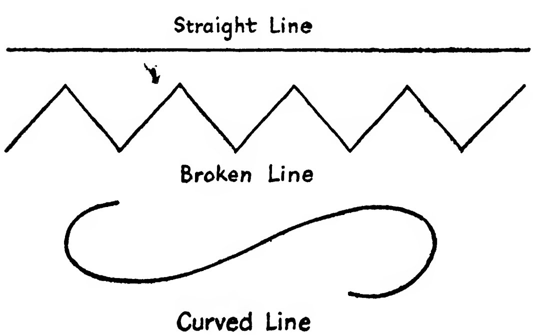
# PRACTICAL GEOMETRY

*William L. Schaaf*

## 13. LINES AND ANGLES

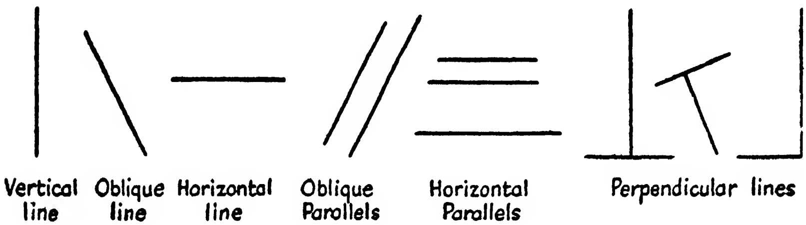
**Nature of Geometry.** The subject of geometry may be studied in two different ways—either as a system of logical demonstrations, or as a body of practical facts and relationships concerning geometric figures and forms. In this book geometry will be presented from the second point of view, since it is the practical applications of geometric principles with which the mechanic and vocational student needs to be familiar. Scarcely a single shop operation or trade process will be encountered which does not involve some geometric relationship or formula. These basic geometric relations can readily be learned without the conventional logical treatment. Most of them are simple to understand on the basis of common sense (intuition) and practical measurements—especially if approached through everyday illustrations from shop problems.

**Lines and Points.** Geometrically we think of a *point* simply as representing a certain position only; it has no magnitude. A *line* may be regarded as the path of a moving point, or the series of positions traced by a point in motion; it has length only—no width or thickness. Lines are either straight, curved, or broken, or a combination of these. A *straight* line may be thought of as an "imaginary" string stretched as tightly as possible (taut); a straight line may also be represented by the intersection of two plane surfaces, such as the edge of a



cube. Any line that is not straight is *curved,* unless, of course, it consists of a series of connected straight lines, i.e., a *broken* line.

If the length of a straight line is not specified, but only its direction, it is to be thought of as extending indefinitely in both directions. If, on the other hand, its terminal, or end points, are specified, it is more properly called a *line segment,* or simply a segment. The various positions in which

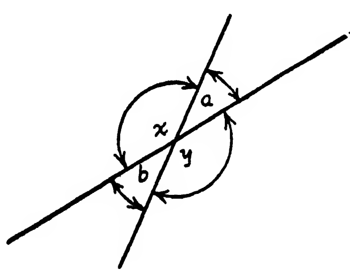



straight lines may lie are indicated by familiar names, as suggested in the accompanying figure. If the word "line" is used without an adjective before it, the reader is to assume that a straight line is meant.

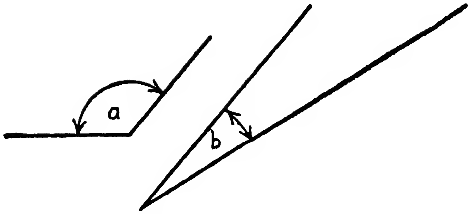
*Parallel* lines are two or more lines, all lying in the same plane surface, and which never meet however far they are extended (prolonged) in either direction. Any two parallel lines are equally distant from each other throughout their entire length; in a series of three or more parallels, however, some may be closer to one another than others.

*Perpendicular* lines are lines forming right angles with each other, where by a right angle is meant a quarter of a complete rotation. A vertical line (plumb line) meeting a horizontal line is an illustration of two perpendiculars; perpendicular lines need not, however, be horizontal and vertical, as shown above. When two lines form right angles with each other, each line is said to be perpendicular to the other.

It simply remains to point out that the shortest distance between any two specified points is a straight line joining those points. Also, that two straight lines can intersect in one point only; indeed, a point might be described as the *position* where two lines intersect. When two straight lines intersect, two pairs of equal angles are formed; the opposite angles, called *vertical angles,* are equal to each other; i.e., $\angle a = \angle b$, and $\angle x = \angle y$.



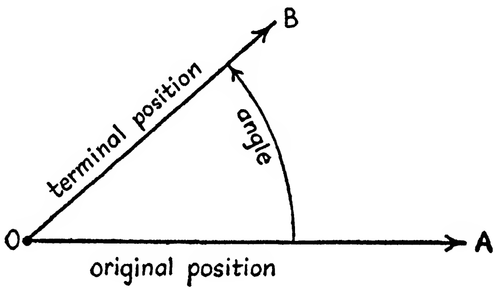

**Angular Measurement.** An *angle* may be described as the "opening" between two lines that intersect or meet. The lines are called the *sides* of




Angle a is larger than angle b, even though its sides are shorter.

the angle, and the point where they intersect or meet is called the *vertex* of the angle. The *size* of the angle in no way depends upon the length of its sides, but only upon the "amount" of opening or divergence between them.

Another way of thinking of an angle is as follows. If a line is rotated about any fixed point on the line, thus taking the line out of its original position to a new terminal position, the two positions are said to form an angle with each other; the point around which the rotation took place is the vertex of the angle. In other words, an angle is really a certain amount of *rotation* or turning. Thus an angle cannot be said to have length, or width, or area, etc., but simply rotation.



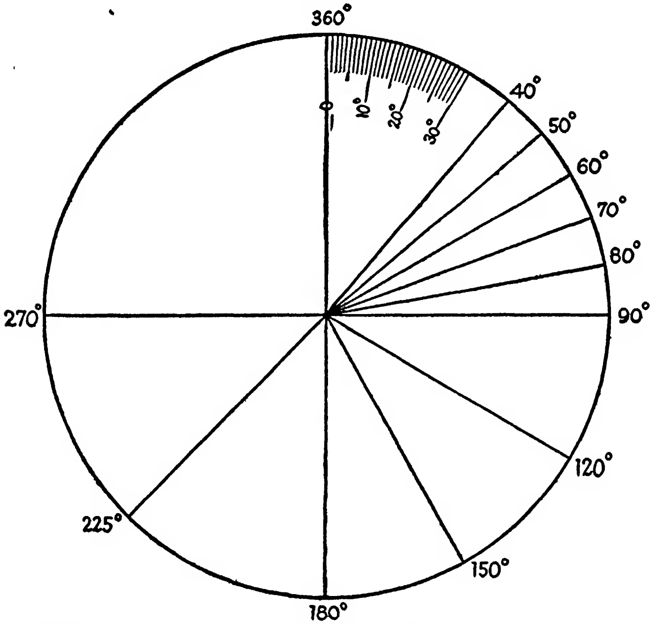

If a line is rotated until it returns to its original position, i.e., through one complete revolution, it is said to have described or formed a *round angle,* or to have turned through 360 degrees. If we consider any particular point on a line, except the pivotal or turning point, it will have described a *circle* when the line has made one complete revolution. Thus there are 360 degrees of angle in every circle, regardless of the size of the circle. For convenience, the standard unit of angular measurement is the *degree,* or $\frac{1}{360}$ of a complete revolution.

**Degrees, Minutes and Seconds.** Angles are measured in terms of the following units:

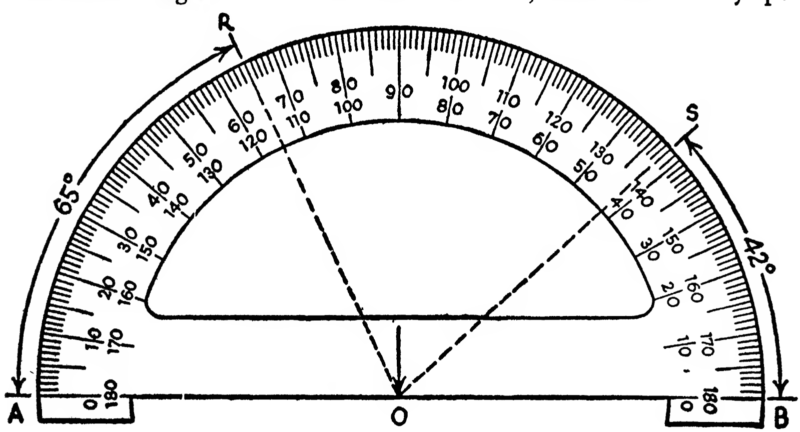
TABLE OF ANGULAR MEASUREMENT

| | |
|---|---|
| 60 seconds | =1 minute |
| 60 minutes | =1 degree |
| 90 degrees | =1 quadrant |
| 360 degrees | =1 circumference |

Degrees are designated by (°); minutes by (′); and seconds by (″); thus an angle of 32 degrees, 45 minutes, 20 seconds is written as 32°45′20″. The accompanying diagram shows a circle divided into degrees; each of the smallest subdivisions represents 1°.

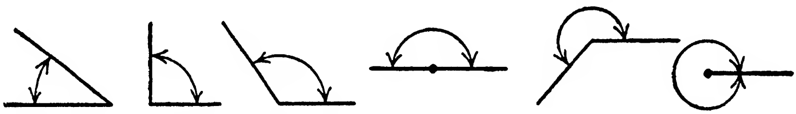



**The Protractor.** This is the instrument commonly used to lay out or measure angles. There are two sets of scales, each from 0° to 180°, one running clockwise from A to B, the other counterclockwise from B to A. Whenever an angle is to be measured or laid out, the vertex is always put



at point 0 on the protractor. The figure shows how an angle AOR (65°) and another angle BOS (42°) can be measured or laid out with the aid of a protractor.

**Kinds of Angles.** A *right* angle is ¼ of a revolution, or 90°. An *acute* angle is an angle less than a right angle, or one containing less than 90°; an *obtuse* angle is one that is greater than a right angle, or containing

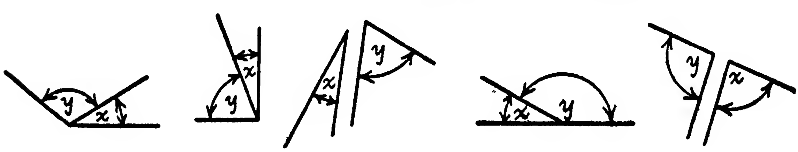


Acute Angle (a) | Right Angle (b) | Obtuse Angle (c) | Straight Angle (d) | Reflex Angle (e) | Round Angle (f)

more than 90° (but less than 180°). If a line has been rotated through *half* a revolution, or 180°, it is said to form a straight angle; thus any straight line may be regarded as a "straight angle" with its vertex at any point on the line we may wish to choose. Again, if a line is rotated through more than 180° but less than 360°, it is said to form a *reflex* angle; however, when two lines form angles as in (e), *unless otherwise specified,* "the angle" between them is the angle *less than* 180°, rather than the reflex angle ("reflex" means, literally, "bending back"). Finally, a complete revolution of 360° is sometimes called a *round angle;* similarly a line making two successive complete revolutions is said to have described an angle of 720°; three revolutions, 1080°; 1½ revolutions, 540°; etc.

**Related Angles.** A few common relations between angles should be understood. Vertical angles, always occurring in pairs, have already been mentioned; such angles are equal to each other (oppositely) in pairs. Other important relations are as follows:

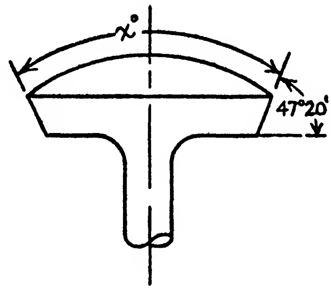
(1) All right angles are equal.
(2) A straight angle equals two right angles.
(3) Two angles are *adjacent* if they have a side in common.
(4) Two angles are *complementary* if their sum equals 90°. Each is the complement of the other; they may or may not be adjacent.
(5) Two angles are *supplementary* if their sum equals 180°. Each is the supplement of the other; they need not be adjacent.



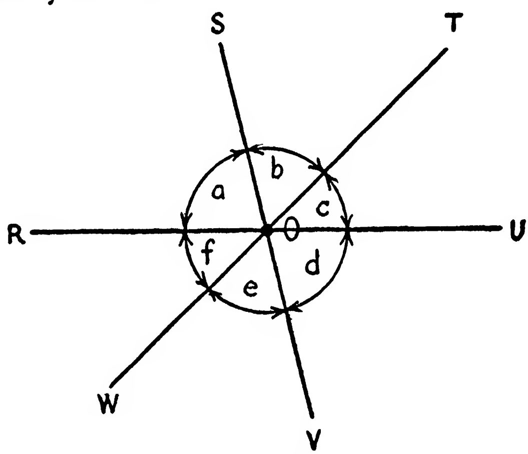

Adjacent Angles | Complementary Angles | Supplementary Angles

### *Exercise 55.*

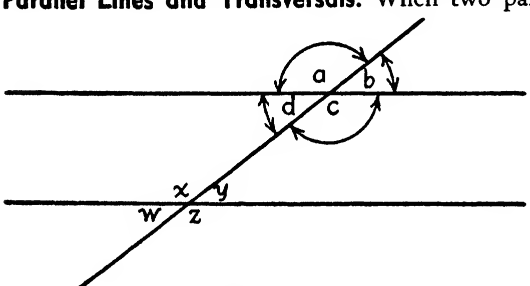
1. What is the complement of 39°? of 22½°? of 44°20′? of 67°41′12″?
2. What is the supplement of 32°? of 45°? of 90°? of 122°? of 57°29′? of 160°44′48″?




3. Find the angle *x* of the valve seat of the gasoline engine valve shown.
4. If ∠ROT=140°, and ∠TOV=120°, find the number of degrees in ∠SOU.
5. If ∠WOS=140°, and ∠ROT=125°, find ∠*d*.
6. Two angles are complementary. The greater exceeds the less by 22°. Find the angles.
7. Find the number of degrees in an angle the sum of whose supplement and complement is 202°.
8. The supplement of a certain angle exceeds three times its complement by 10°. Find the angle.




**Parallel Lines and Transversals.** When two parallel lines are cut by a third line, the latter, which is said to be a *transversal,* forms four pairs of angles with the two parallels. These are known as follows:



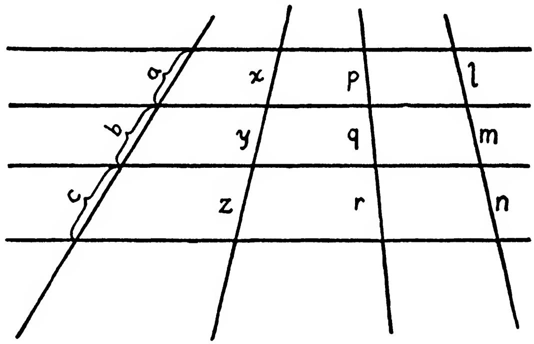

$$\left.\begin{array}{l} \angle a \text{ and } \angle x \\ \angle b \text{ and } \angle y \\ \angle c \text{ and } \angle z \\ \angle d \text{ and } \angle w \end{array}\right\} \text{are } \textit{corresponding} \text{ angles.}$$

$\left.\begin{array}{l}\angle d \text{ and } \angle y\\ \angle c \text{ and } \angle x\end{array}\right\}$ are *alternate-interior* angles.

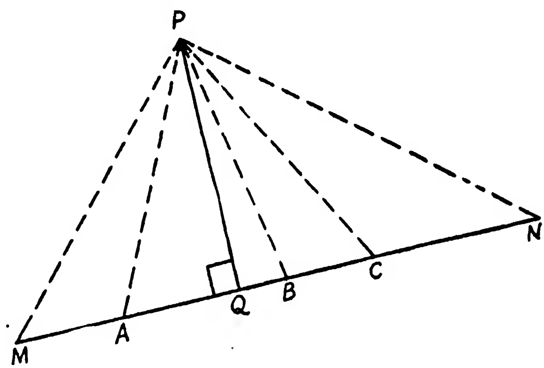
$\left.\begin{array}{l}\angle a \text{ and } \angle z\\ \angle b \text{ and } \angle w\end{array}\right\}$ are *alternate-exterior* angles.

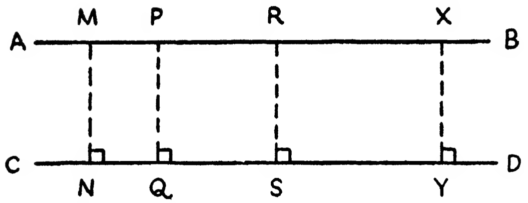
By studying the figure carefully, it will be seen that whenever a transversal intersects a pair of parallel lines, all the pairs of corresponding angles, as well as all the pairs of alternate angles, whether "interior" or "exterior," are equal. Thus: $\angle a = \angle x$; $\angle d = \angle y$; $\angle a = \angle z$; etc. Angles that lie on the *same* side of the transversal and are *included* between the parallels, are not equal, but *supplementary* to each other; thus $\angle d + \angle x = 180°$, and $\angle c + \angle y = 180°$.

If a series of parallel lines is cut by two or more transversals, and the segments cut off on any one of these transversals are equal to each other, then the segments on every other transversal are also equal to each other. On the diagram, this means that if $a=b=c$, then $x=y=z$; $p=q=r$; and $l=m=n$. It does *not* mean, however, that $a=x=p=l$, or that $b=y=q=m$, etc.; whether this is true depends upon the angles at which any two transversals cut the parallels.



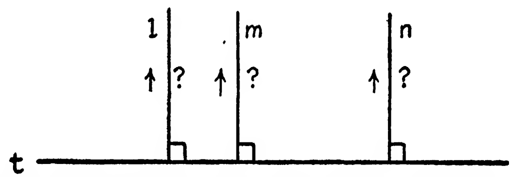

**Meaning of Distance.** In geometry, when we speak of the *distance* from a point to a line (or from a line to a point) it is understood to mean the *shortest* distance, or the *perpendicular* drawn from the point to the line. Thus, while PM, PA, PB, PC, and PN do represent "distances" from P *to various points on MN,* the distance from P *to the line MN* is PQ, and no other. Similarly, the distance between two parallel lines refers to the *perpendicular* distance be-



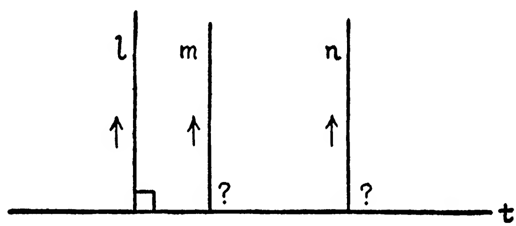



tween them, anywhere along the line; this perpendicular distance is the same at any point, since two parallel lines are everywhere equidistant.

**Further Properties of Parallels and Perpendiculars.** The following relationships are not only fundamental, but frequently quite useful as well.

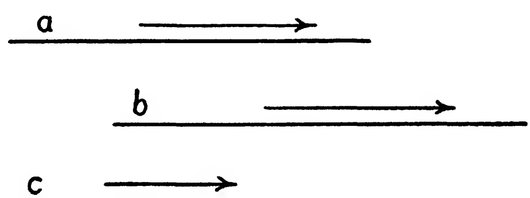



(1) If two or more lines are all perpendicular to another line, then they are parallel to one another. For example, if *l, m,* and *n* are each $\perp$ *t,* then *l, m* and *n* are $\parallel$ each other.




(2) If any one of two or more parallel lines is perpendicular to another line, then all of them are perpendicular to that line. For example, if *l, m* and *n* are $\parallel$ each other, and $l \perp t$, then *m* and *n* are also $\perp$ *t.*

(3) If a line is perpendicular to one of two or more parallels, then it is perpendicular to all of them. For example, if *l, m,* and *n* are $\parallel$, and if $t \perp l$, then *t* is also $\perp$ *m* and *n.*

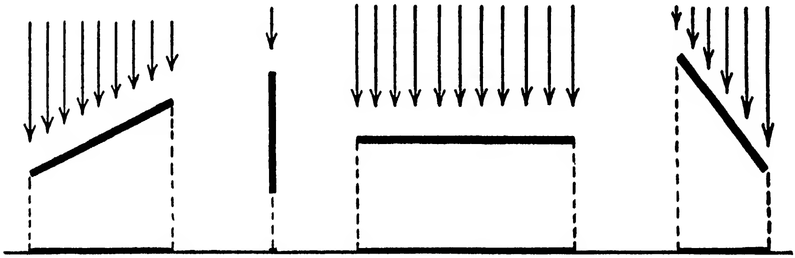



(4) If two lines are both parallel to a third line, then they are parallel to each other. For example, if $a \parallel x$, and $b \parallel x$, then $a \parallel b$.

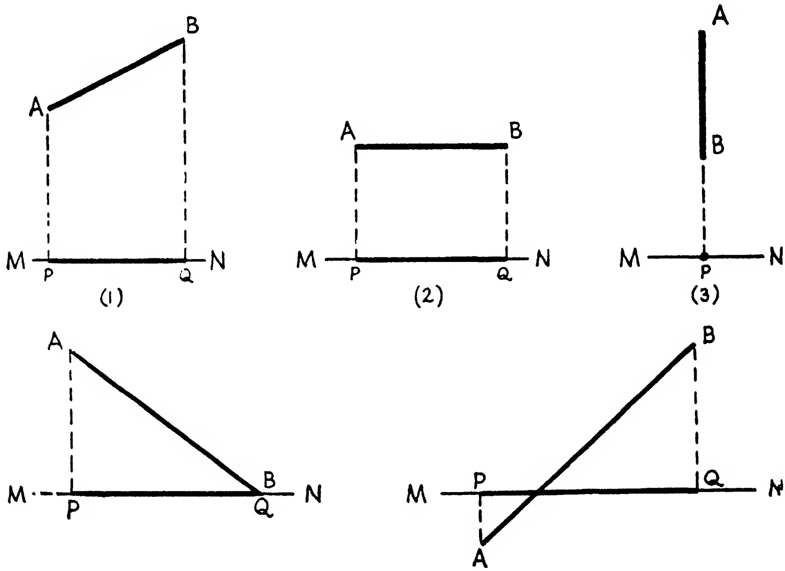
(5) If a line is parallel to one of two other parallel lines, then it is parallel to the other. For example, if $a \parallel b$, and $x \parallel a$, then *x* is also $\parallel$ *b.*

**Projections.** Everybody knows that when the sun is directly overhead, the shadow cast by a stick held obliquely will be shorter than the stick. The more nearly vertically it is held, the shorter the shadow becomes;

when it is exactly vertical, its shadow is shortest and approximates a point. The more nearly horizontal its position, the longer its shadow be-

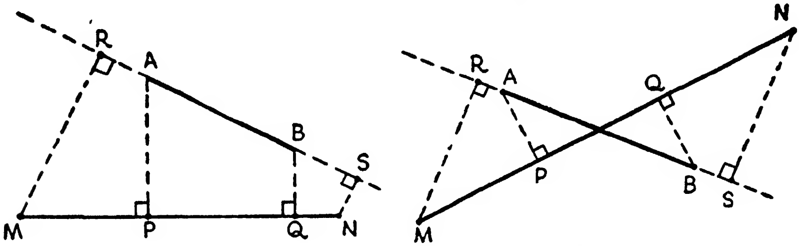


comes; when exactly horizontal, the shadow is longest, being equal to the length of the stick itself. In all these cases, the shadow of the stick is called its *projection* (technically, its orthographic projection). Or, expressed somewhat differently, the projection of one line upon a second line is the segment included between the feet of the perpendiculars drawn to the second line from the extremities of the first line. This is illustrated in the figures below:




In each case, the projection of line AB upon line MN is the segment PQ; even in case (3), where the projection is the point P, this point may be regarded as a segment PQ of zero length (the points P and Q having "come together," or coincided).

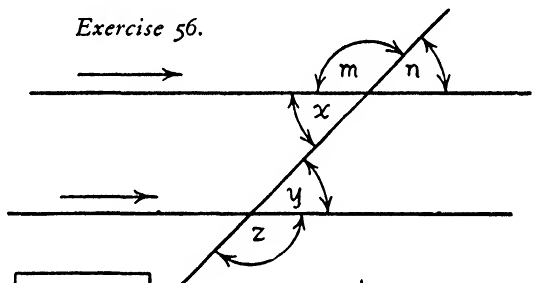
Furthermore, given any two lines, either line may be projected upon the other, although one of them may have to be prolonged first. Thus in



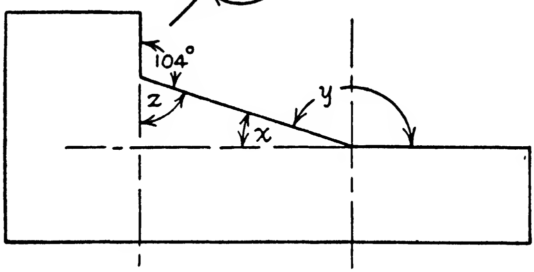

(1) (2)

(1), the projection of AB upon MN is the segment PQ, while the projection of MN upon AB is the segment RS (where AB was first extended in both directions). Similarly in (2), the projection of AB upon MN is PQ; and the projection of MN upon AB is RS.

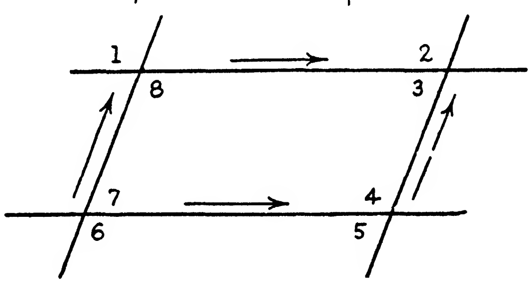
*Exercise 56.*




1. (a) If $\angle x = 38°$, find $\angle z$.
   (b) If $\angle m = 112°$, find $\angle y$.
   (c) If $\angle z = 155°$, find $\angle n$.

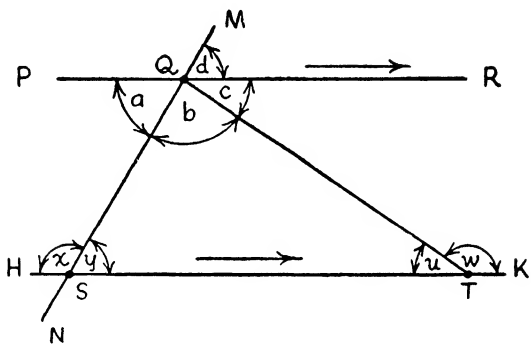



2. In the metal plate shown, find the number of degrees in each of the angles $x$, $y$ and $z$.

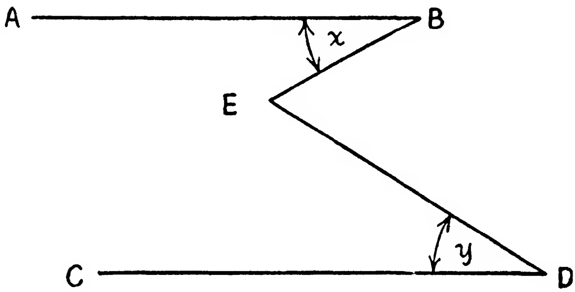



3. (a) If $\angle 1 = 120°$, find $\angle 4$.
   (b) If $\angle 7 = 58°$, find $\angle 3$.
   (c) If $\angle 8 = 115°$, find the sum of $\angle 3 + \angle 4 + \angle 7 + \angle 8$.

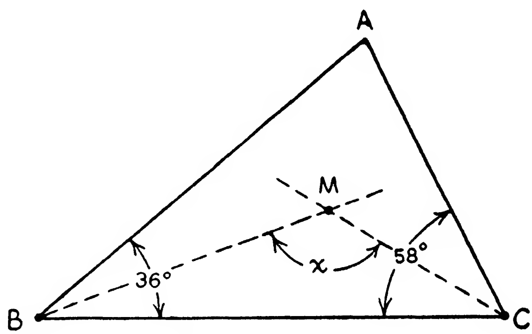
4. (a) If $\angle y = 50°$, and $\angle u = 65°$, find $\angle MQT$.
(b) If $\angle y = 70°$, and $\angle u = 42°$, find $\angle b$.
(c) If $\angle b = 85°$, and $\angle u = 45°$, find $\angle x$.
(d) If $\angle a = 48°$, and $\angle w = 160°$, find $\angle TQM$; also $\angle PQM$.



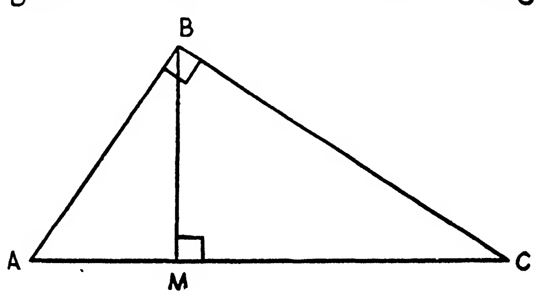

5. Line AB is parallel to CD; if $\angle x = 26°30'$, and $\angle y = 38°15'$, how many degrees are there in $\angle$ BED? (*Hint:* Draw a line through E, parallel to AB.)




6. If two angles of a triangle are 58° and 36°, respectively, what is the angle ($x$) formed by their bisectors?

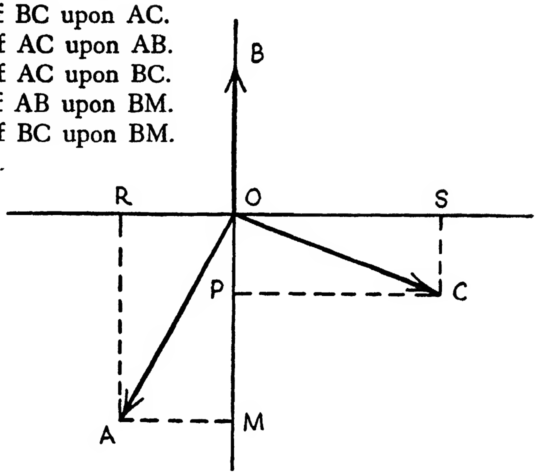



7. If $\angle ABC$ is a right angle, and $\angle AMB$ is also 90°, name the following:
(a) the projection of AB upon AC.



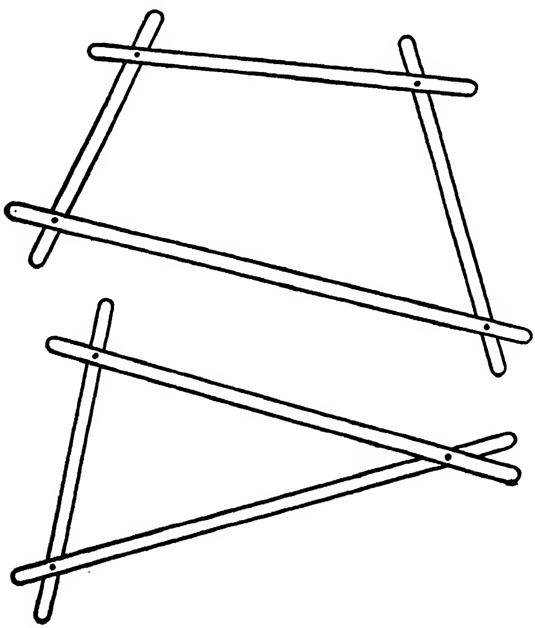
(b) the projection of BC upon AC.
(c) the projection of AC upon AB.
(d) the projection of AC upon BC.
(e) the projection of AB upon BM.
(f) the projection of BC upon BM.

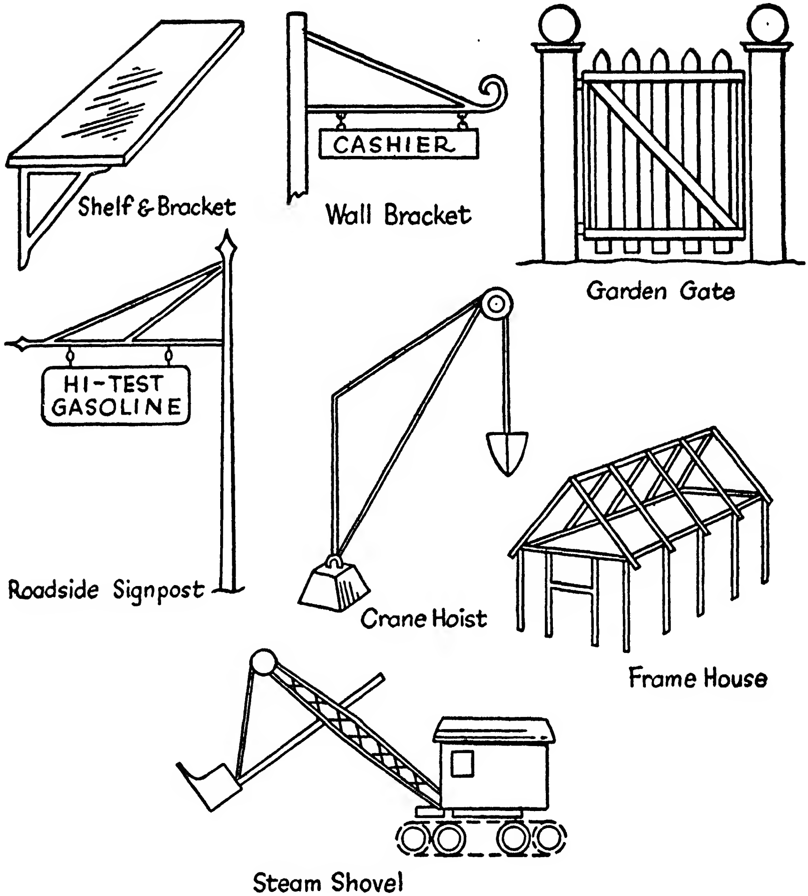
8. In this diagram of forces, state (a) the projection of OA, OB and OC, respectively, upon the horizontal axis; (b) the projections of OA, OB and OC, respectively, upon the vertical axis.



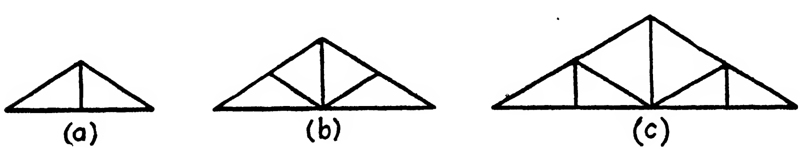

## 14. TRIANGLES AND POLYGONS

**Rigid Frames.** If four strips of wood are pinned together with nails, everyone knows that such a frame can easily be deformed, i.e., changed in shape. But if only *three* sticks be similarly fastened together, the resulting triangular frame cannot be changed in shape. In other words, triangles, or three-sided figures, are rigid frames; their *shape* cannot be altered if the lengths of the sides are fixed. For this reason triangular frames are used in structures to secure greater strength and rigidity. In fact, frame structures and trusses are commonly used in engi-

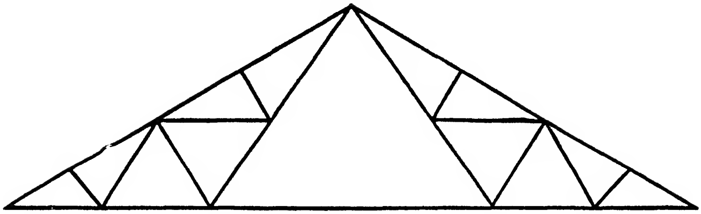




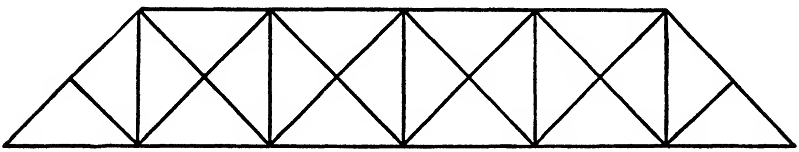

neering and structural work, such as bridges, roofs, pillars, towers, and supporting members and frameworks. A few types of trusses are shown below.




Simple Trusses

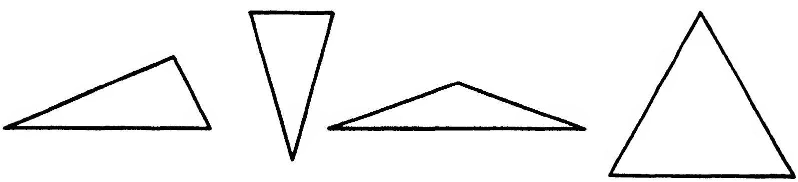


Roof Truss



Railroad Bridge

**Kinds of Triangles.** A triangle, then, is a straight-line figure having three sides and three angles. The vertices of the angles are also called the vertices of the triangle. If the lengths of the sides have been chosen, the "shape," or angles of the triangle, are automatically determined (fixed). Triangles may be classified according to the lengths of their sides, as follows:



(1) Scalene Triangle (2) Isosceles Triangles (3) Equilateral triangle

In an *equilateral* triangle, all three sides are equal in length.

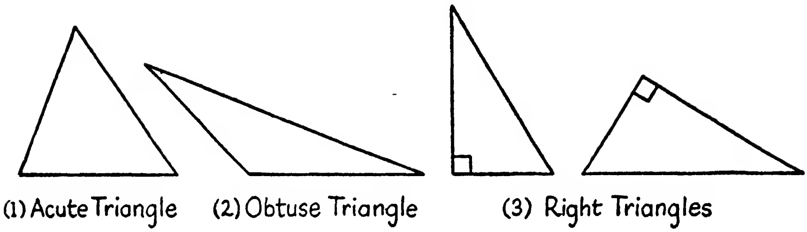
In an *isosceles* triangle, only two sides are equal in length.

In a *scalene* triangle, no two sides are equal in length. In an isosceles triangle, the two equal sides are called the *arms,* and the third remaining side is called the *base.* An equilateral triangle is obviously also isosceles; any one of its sides may be regarded as the base.

Triangles are also classified according to their angles, as follows:

In an *acute* triangle, each of the three angles is less than 90°.

In an *obtuse* triangle, one of the three angles is obtuse, and the two



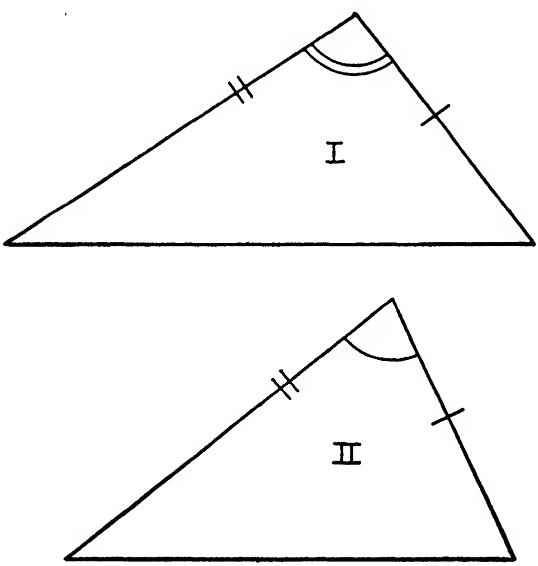

remaining angles are both acute. A triangle cannot have more than one obtuse angle.

In a *right* triangle, one of the angles is a right angle, and the two remaining angles are acute angles. A triangle cannot have more than one right angle.

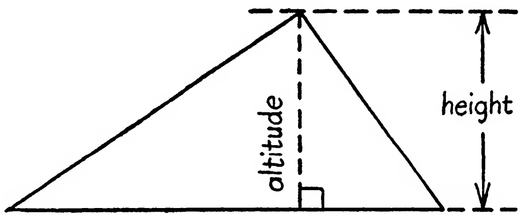
The sum of any two sides of a triangle is always greater than the third side, since a straight line is the shortest distance between any two given points.

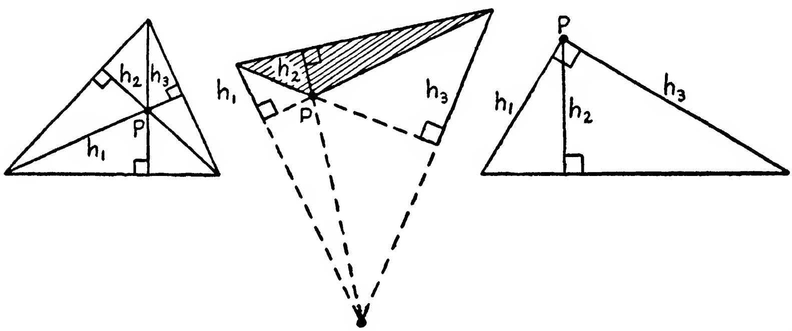
If the angles of a triangle are unequal, the longest side is opposite the largest angle, and *vice versa;* also the shortest side is opposite the smallest angle, and *vice versa.* In a right triangle, the side opposite the right angle is thus the longest side; it is known as the *hypotenuse.*

If two sides of one triangle are respectively equal to two sides of another, but the included angle of the first is greater than the included angle of the second, then the third side of the first triangle is longer than the third side of the second triangle. Conversely, if two pairs of sides are respectively equal, but the third side of one triangle is longer than the third side of a second triangle, then the angle opposite the longer side is greater than the angle opposite the shorter side.



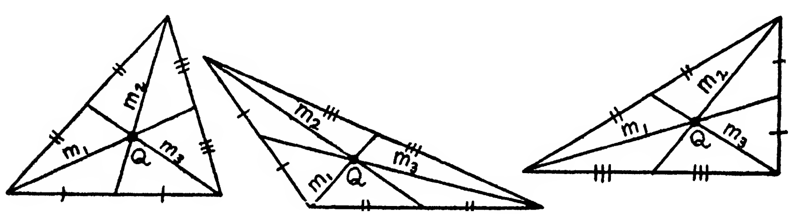
**Altitudes and Medians.** An *altitude* of a triangle is the perpendicular distance from any vertex to the opposite side. Every triangle therefore has three altitudes, one from each of the three vertices. In an acute triangle, all three altitudes fall *inside* the triangle; in an obtuse triangle, two of the altitudes fall *outside* the triangle; and in a right triangle, two of the altitudes coincide with the two sides.



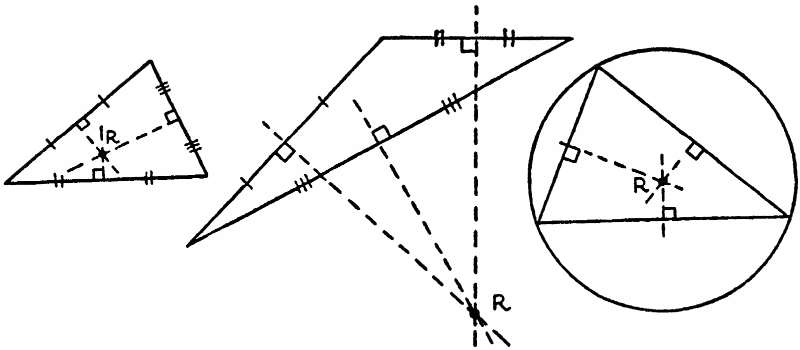




All three altitudes of a triangle (prolonged, if necessary) meet in the same point; this point of intersection (P) lies within, without, or on the triangle, according as the triangle is an acute, an obtuse, or a right triangle, respectively.

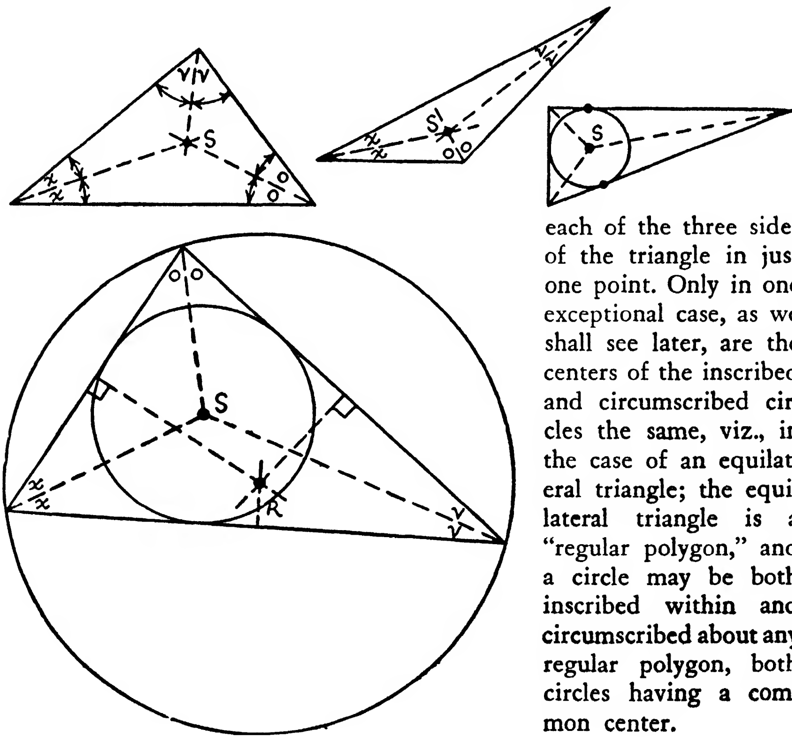
A *median* of a triangle is a line drawn from any vertex to the midpoint of the opposite side. Every triangle has three medians, which must of necessity lie entirely within the triangle. All three medians of a triangle meet in the same point. This point of intersection (Q) divides each median into two segments which are in the ratio of 2:1, respectively; i.e., the shorter segment of the median is, in each case, ⅓ the length of that entire median.



**Bisectors of the Sides and Angles.** The perpendicular bisectors of the sides of a triangle also meet in a single point. This point is equally distant from the three vertices; hence if it is used as a center, a circumscribed circle can be drawn which will pass through all three vertices.

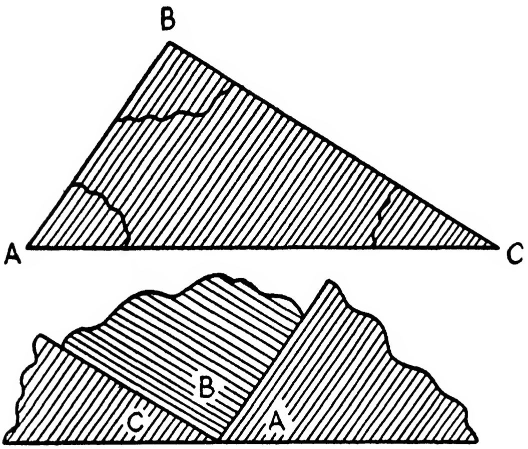


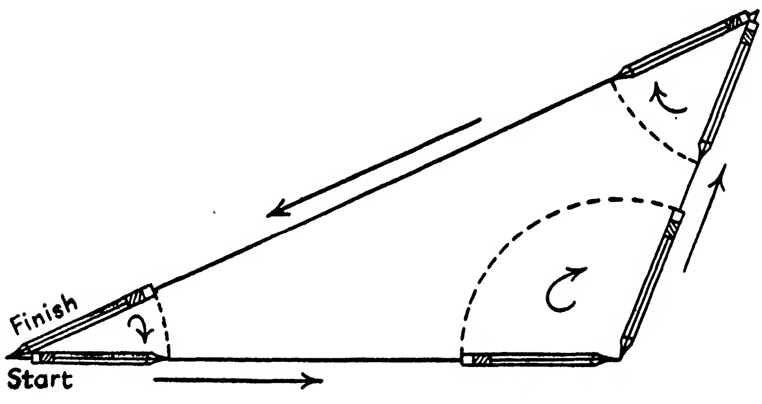

Similarly, the bisectors of the angles of a triangle also meet in a point, but this point is equally distant from all three sides of the triangle. Hence it is the center of the circle *inscribed* in the triangle, i.e., touching




each of the three sides of the triangle in just one point. Only in one exceptional case, as we shall see later, are the centers of the inscribed and circumscribed circles the same, viz., in the case of an equilateral triangle; the equilateral triangle is a "regular polygon," and a circle may be both inscribed within and circumscribed about any regular polygon, both circles having a common center.

**Sum of the Angles of a Triangle.** One of the most important properties of a triangle is the fact that, irrespective of its shape, the sum of the three angles always equals 180°. This may be seen by tearing off the three "corners" of a triangular piece of cardboard and fitting the pieces together as shown; when rearranged, the three angles taken together form a straight angle, or 180°. Another way of showing that the angle sum equals 180° is by tracing the sides of a triangle, by sliding and rotating a pencil as suggested in the accompanying sketch. When the triangle has been completely traversed, the pencil will be found to have been reversed in direction; in other words, it has been turned through half a



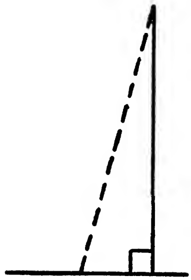

revolution, or 180°, again showing that the sum of the three angles of the triangle equals a straight angle.

Many properties of geometric figures depend upon this fact. Some of these are:

(1) That no triangle may have more than one right angle or one obtuse angle.

(2) That only one perpendicular may be drawn to a line from a point outside.

(3) That the acute angles of a right triangle are complementary.



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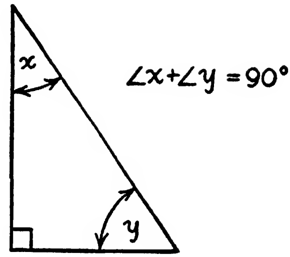
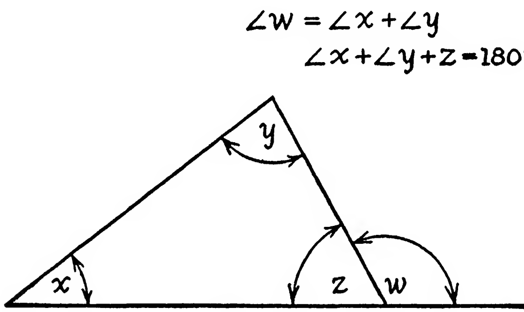


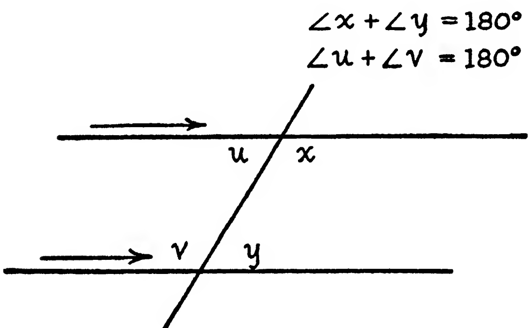

(3)

(4) That an exterior angle of a triangle (formed by prolonging any side) is equal to the sum of the two remote interior angles.

(5) That angles included between parallels, and on the same side of a transversal, are supplementary.




(4)

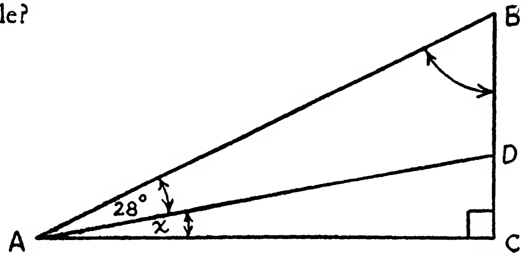



(5)

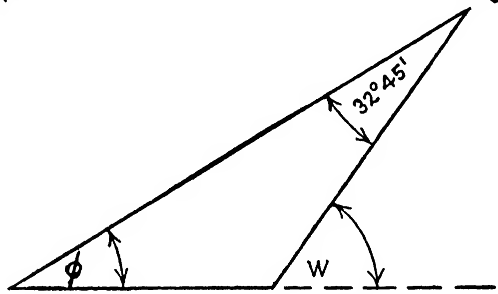
*Exercise 57.*

1. If one acute angle of a right triangle equals 14°36′, what is the value of the other acute angle?

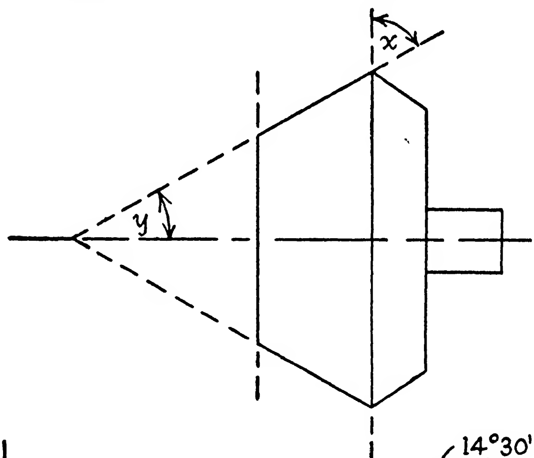
2. In the right triangle ABC, find $\angle x$ if $\angle B$ equals 46°.



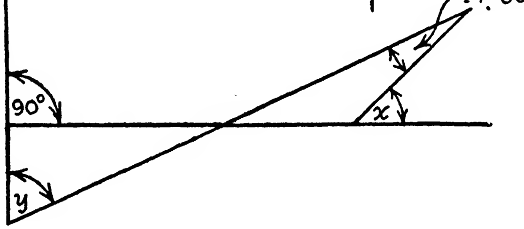

3. If $\angle w = 78°12'$, find the value of $\angle \phi$.



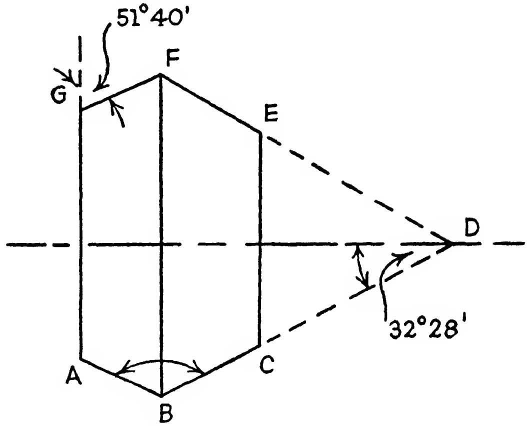

4. In this sketch of a bevel gear blank, if $\angle y = 61°24'\,10''$, find $\angle x$.



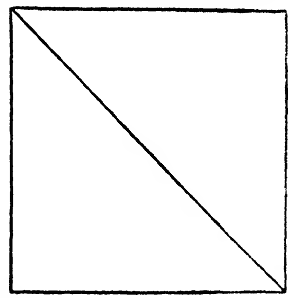

5. Find the value of $\angle x$, if $\angle y$ equals 39°52′.



6. Find $\angle ABC$, if the face angle at G is $51°40'$ and the half-angle at D equals $32°28'$.

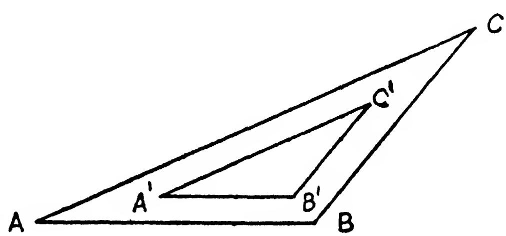



**Congruence, Similarity, and Equivalence.** If two triangles can be made to coincide or "fit exactly," they are said to be *congruent.* Congruent triangles might be said to have the same "shape" and the same "size." Any two triangles are congruent if:

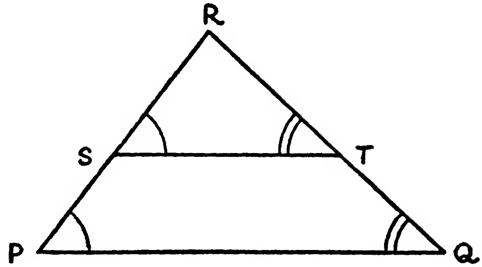


(1) two sides and the included angle of one triangle are equal respectively to two sides and the included angle of the other;
(2) two angles and the included side of one triangle are equal respectively to two angles and the included side of the other;
(3) if a side and any two angles of one triangle are equal respectively to a side and the corresponding angles of the other;
(4) if all three sides of one triangle are respectively equal to the three sides of the other (even though nothing is known about the angles);
(5) if they are right triangles, and the hypotenuse and one side of one triangle are equal respectively to the hypotenuse and corresponding side of the other.

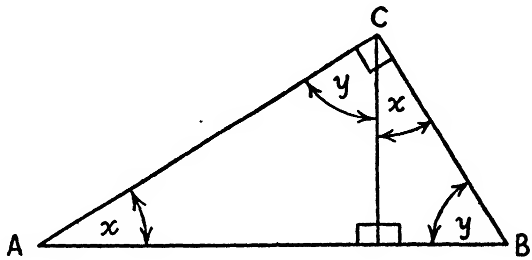
Two triangles are not necessarily congruent, however, merely because the three angles of the one are equal respectively to the three angles of the other; in this case they are said to be *similar*



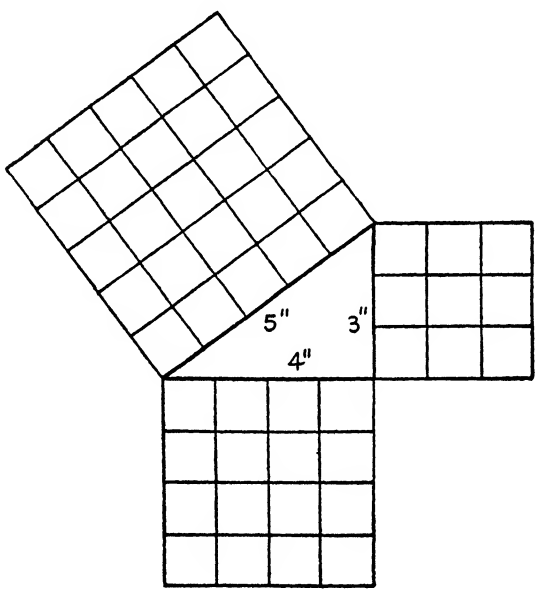
triangles. Such triangles have the same "shape," but not the same "size." We shall learn more about similar triangles later.




Triangles which are neither congruent nor similar may nevertheless be equivalent in area (i.e., cover the same surface), just as, for example, rectangle 4″ ×9″ covers the same surface as a square 6″ ×6″; the rectangle and the square are not congruent, neither are they similar in shape, yet they are *equivalent* in area. We shall learn more about this, too, later on.

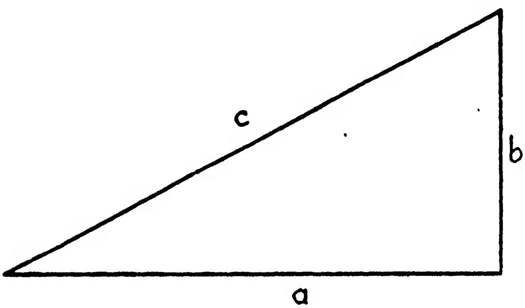



**The Right Triangle Rule.** One of the most famous and useful geometric relations is the relation between the lengths of the three sides of a right




Since $3^2+4^2=5^2$, or $9+16=25$, it is clear that if we *square* the number of units in each side and add these results, the sum obtained will be the square of the number of units in the hypotenuse.

triangle, viz., that $a^2+b^2=c^2$. Put into words: the square of the hypotenuse equals the sum of the squares of the other two sides. By means of this relation we can find the length of any side of a right triangle if we know the lengths of the other two sides. We simply use one of the following formulas:




$c^2=a^2+b^2$ or $c=\sqrt{a^2+b^2}$
$a^2=c^2-b^2$ or $a=\sqrt{c^2-b^2}$
$b^2=c^2-a^2$ or $b=\sqrt{c^2-a^2}$

*Rule* 1. To find the hypotenuse of a right triangle, find the square root of the sum of the squares of the other two sides.

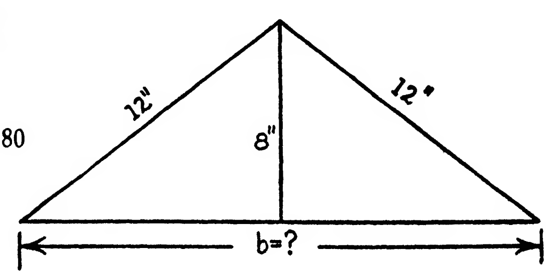
*Rule* 2. To find either side of a right triangle, if the other side and the hypotenuse are known, find the square root of the difference between the square of the hypotenuse and the square of the known side.

Example 1: Find the length of the base of the isosceles triangle shown.

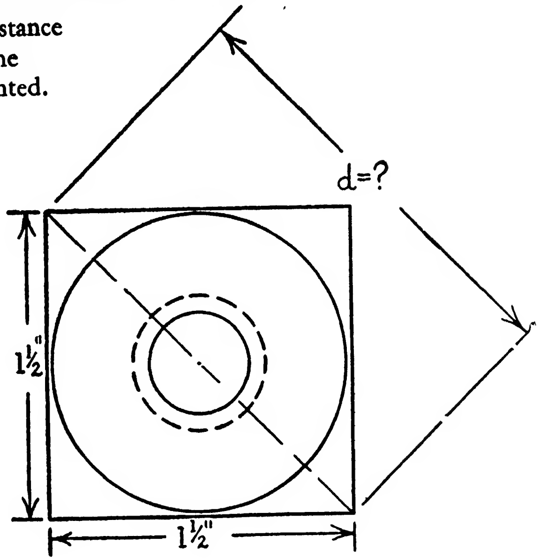
Solution: The altitude, 8″, divides the base in half. Call the half-base $x$.

$x^2+8^2=12^2$
$x^2=12^2-8^2=144-64=80$
$x^2=80$
$x=\sqrt{80}=8.94+$,

Therefore the base $= 2\times 8.94=17.9''$, *Ans.*



EXAMPLE 2: Find the distance across the corners of the squarehead nut represented.




SOLUTION:

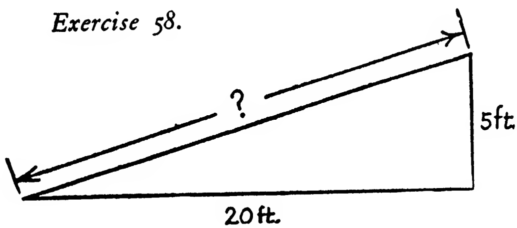
$d^2=(3/2)^2+(3/2)^2$

$d^2=9/4+9/4=18/4=4.5$

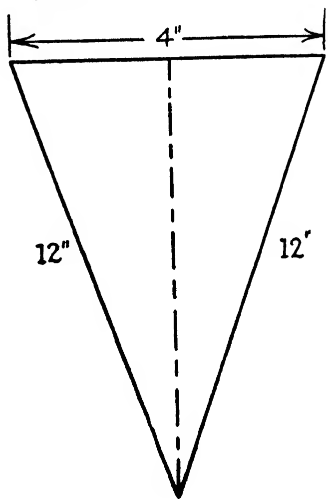
$d=\sqrt{4.5}=2.1''$, *Ans.*

*Exercise 58.*

1. If the base of an isosceles triangle is 16″ and the altitude is 9″, find the length of the equal sides.



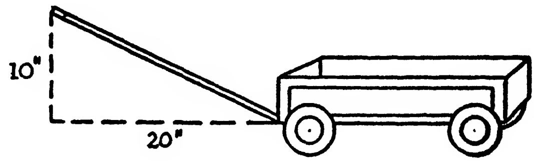

2. The runway of a garage ramp rises 5 ft. in a horizontal distance of 20 ft. Find the length of the ramp.

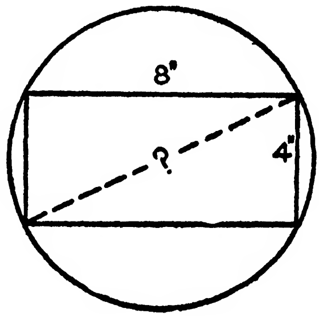



3. A wedge in the shape of an isosceles triangle has the dimensions shown. Find the length of the center line.

4. If the normal lens used in a camera has a focal length equal to the diagonal of the film which the camera accommodates, what should be the focal length of the lens for a camera using 4″×5″ plates?

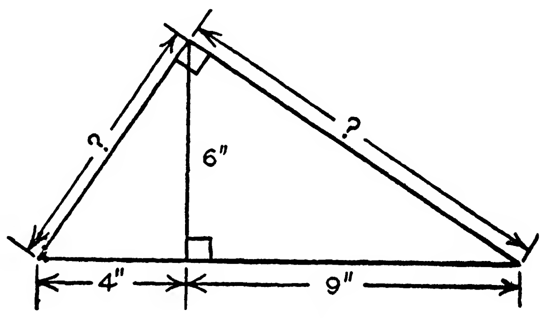
5. A child pulling a cart stands 20″ in front of it. One end of the handle is 10″ higher than the other. How long is the handle?



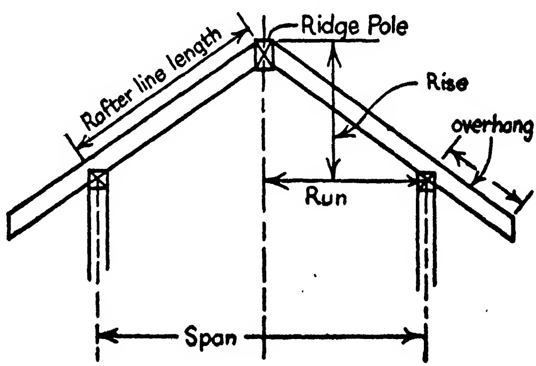




6. A wooden beam 4″×8″ is to be cut from a log. What is the diameter of the smallest log that could be used?

7. The altitude of a right triangle drawn to the hypotenuse is 6″. If the altitude divides the hypotenuse into segments of 4″ and 9″, find the lengths of the other two sides. How could you check your result?

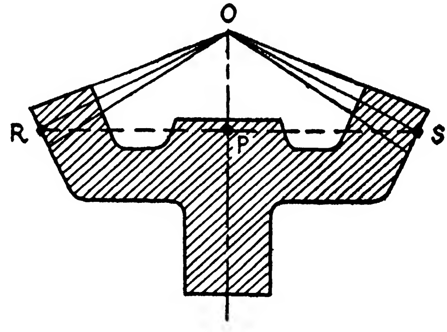



8. With an overhang of 18 in., a rise of 16 ft. and a span of 28 ft., find the overall length of rafter required.

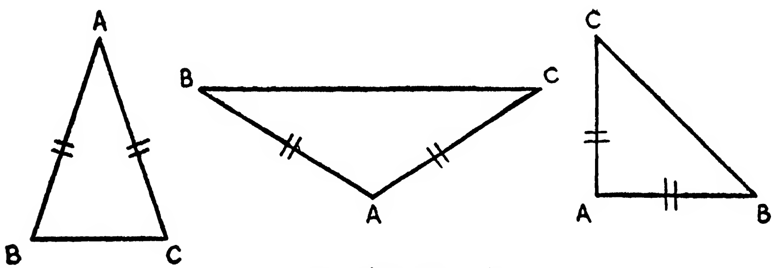


9. A square is inscribed in a circle whose radius is 4″. Find the length of the side of the square.

10. The distance OR in a bevel gear is called the apex distance, and RS is called the pitch diameter. If OP is 2½″ and OR is 5½″, find the pitch diameter.




**Isosceles Triangle.** Any triangle having at least two of its sides of equal length is called an *isosceles* triangle. The third side (BC) is referred to as the *base,* although it need not necessarily be the side on which the




Isosceles Triangle

triangle is "standing." That vertex (A) of the triangle which lies opposite the base is called *the vertex* of the isosceles triangle. The altitude drawn from the vertex to the base has the following properties:

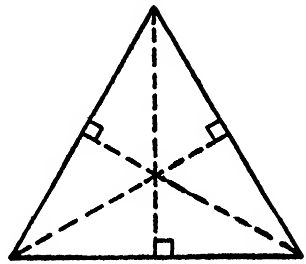
(1) It bisects the base (it is thus a median).
(2) It bisects the angle at the vertex (the "vertex angle").
(3) It divides the isosceles triangle into two congruent triangles.

Another property of the isosceles triangle is that its *base angles* are equal; these are the two angles opposite the equal sides, i.e., adjacent to the base. It is also true that if any two angles of a triangle are equal, then the sides opposite them are equal, i.e., the triangle is isosceles.

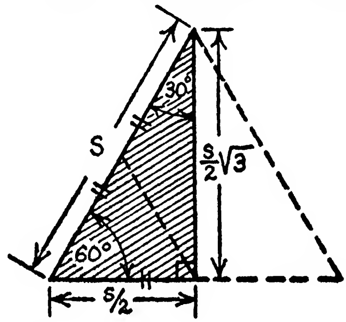
**Special Triangles.** An *equilateral triangle* has special properties, not found in other triangles. Thus its three altitudes are all equal in length;

so are the three medians. In fact, the medians and altitudes are identical with one another; they are also identical with the angle-bisectors and with the perpendicular bisectors of the sides. A general formula for the altitude of an equilateral triangle is

$$h=\frac{s}{2}\sqrt{3}, \text{ where } s=\text{length of a side.}$$

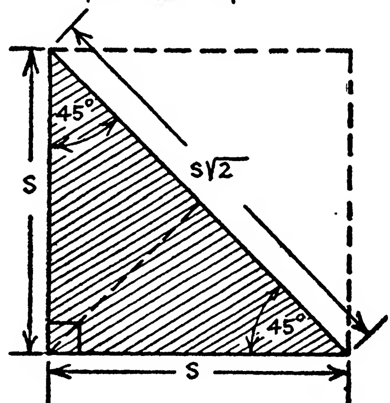


Another "special" triangle is the so-called *30°—60°—90° triangle,* or half an equilateral triangle. It is clear that in such a triangle, the hypotenuse is twice as long as the side opposite the 30° angle, or the shortest side. Moreover, the median drawn from the vertex of the right angle to the hypotenuse equals half the hypotenuse in length.



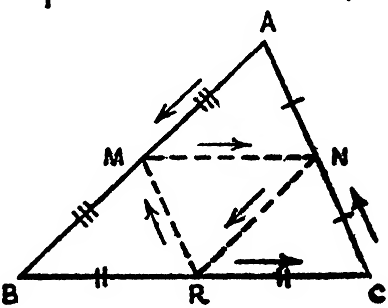

The *isosceles right triangle* is another special triangle often encountered. It is simply half of a square. In an isosceles right triangle, the altitude drawn to the hypotenuse equals

$$\frac{s}{2}\sqrt{2}.$$

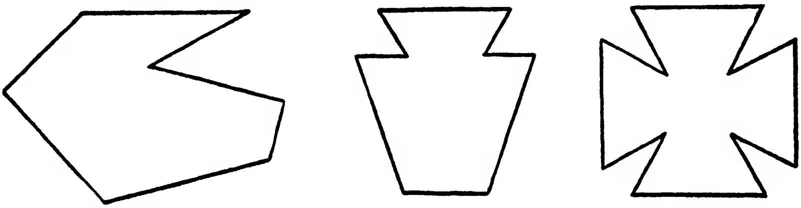



**Mid-join of a Triangle.** A line joining the midpoint of any side of a triangle with the midpoint of any other side is sometimes called the *mid-join* of the triangle. The mid-join has two important properties: (1) it is parallel to the third side of the triangle; (2) it is equal in length to *half* the third side.

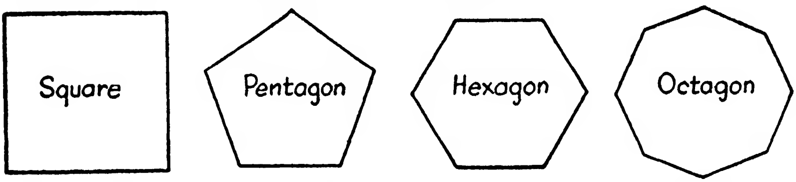
Thus if BM=MA and AN=NC, then MN (the mid-join) is parallel to BC and equal to ½ (BC). If the other two mid-joins be drawn, the original triangle is divided into four congruent triangles and three equivalent parallelograms.



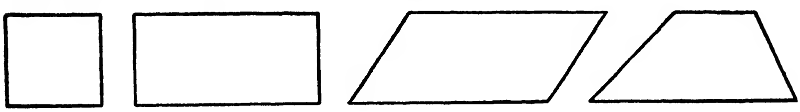
**Polygons.** Any rectilinear figure having four or more sides is called a *polygon.* Polygons may be irregular or regular. Irregular polygons may



be *convex* or *reëntrant.* A *regular* polygon is a convex polygon all of whose sides are equal and all of whose angles are equal. Examples of regular polygons are shown below; both kinds of polygons are frequently encountered in shop practice and trade operations.

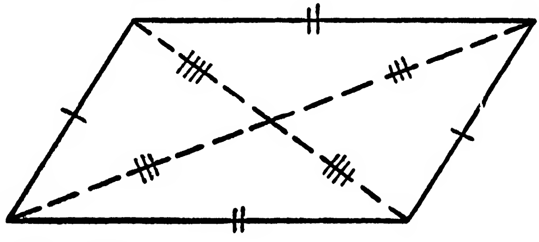



**Quadrilaterals.** Four-sided polygons are called *quadrilaterals.* Such figures are not rigid, as in the case of the triangle. In other words, the shape of a quadrilateral is *not* determined by the lengths of its sides; a figure of four fixed-length sides may be distorted by changing its angles *without* changing the lengths of its sides. The sum of the four angles of every quadrilateral, however, is constant, and equals 360°. Important types of quadrilaterals are shown below.



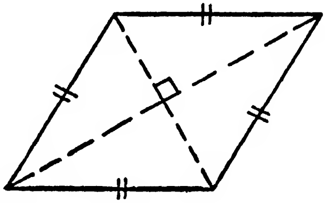
Square Rectangle Parallelogram Trapezoid

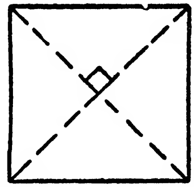
**Parallelograms.** A quadrilateral having its opposite sides parallel in pairs is called a *parallelogram.* Some of the most important properties of all parallelograms are:



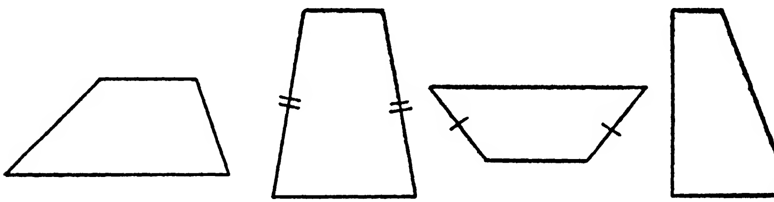
(1) the opposite sides are equal.
(2) the opposite angles are equal.
(3) the diagonals bisect each other.

The *square* and the *rectangle* are special cases of the parallelogram; in both cases, all four angles are equal, each being 90°; the square differs from the rectangle in being equilateral, i.e., all four sides are equal. A parallelogram (other than a square) which is equilateral is called a *rhombus.* A special feature of the rhombus, which does not hold for non-equilateral parallelograms, is the fact that its diagonals not only bisect each other, but intersect at right angles as well, a feature that holds for squares also. The *altitude* of any parallelogram is the perpendicular distance between a pair of bases; either pair of parallel sides may be regarded as bases.





**The Trapezoid.** Any quadrilateral having two and only two of its sides parallel is known as a *trapezoid;* if the remaining two sides are equal in length it is an *isosceles trapezoid.* In either case, the two parallel sides are called the *bases* of the trapezoid. Trapezoidal shapes are frequently encountered in various machine parts and in construction work. The midjoin of a trapezoid, also commonly called the *median* of the trapezoid, is a line joining the mid-points of the two non-parallel sides. The median of a trapezoid is



(1) parallel to the two bases; and
(2) equal to half the sum of the two bases.

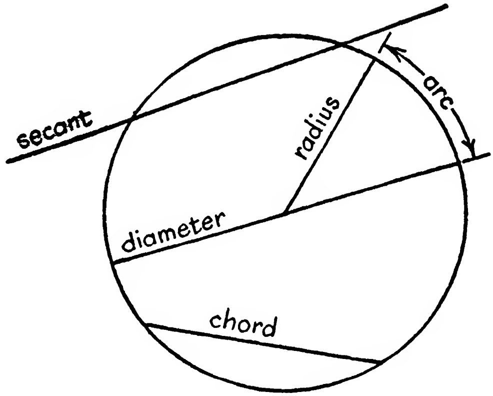
The *altitude* of a trapezoid is the perpendicular distance between the two bases.

## 15. CIRCLES AND TANGENTS

**Circles.** The circle is one of the earliest geometric forms known to primitive man. It was doubtless suggested by the rims of his pottery or the edge of spherical fruit when cut in half.

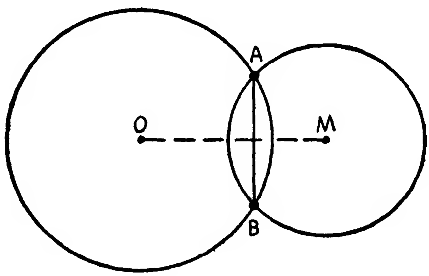
A *circle* is simply a closed line, every point of which is equally distant from a point within called the *center.* The distance from the center to the circle, or a line from the center to any point on the circle, is called the *radius.* Obviously, all the radii of any given circle are equal to one

another. A straight line can intersect a circle in two points and only two; such a line is called a *secant.* If the ends of a line terminate in a circle, the line is called a *chord.* A *diameter* is thus also a chord; it is the longest chord that can be drawn in a circle, and is exactly twice the length of a radius. Naturally, all diameters of the same circle are equal.

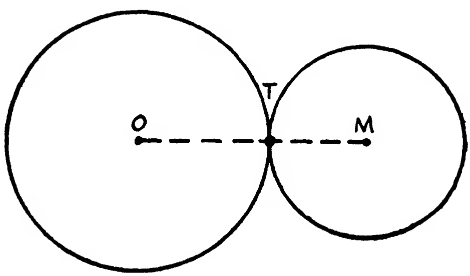



An *arc* of a circle is that part of the circle included between any two particular points. The ends of a diameter divide the circle into two equal arcs. If an arc is less than half the circle, or *semicircle,* it is a *minor* arc; if greater than a semicircle, it is a *major* arc. The straight line joining the ends of an arc is called the *chord* of that arc; a chord is said to subtend the arc. A diameter is thus a chord subtending a semicircle.

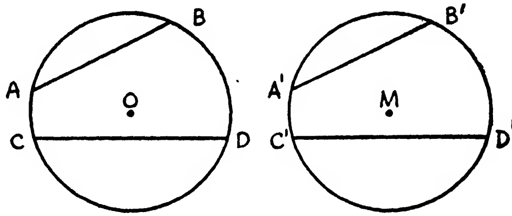
Two circles can intersect each other in only two points; the chord joining these points is called a common chord. Circles may also be *tangent* to each other, which means that they touch each other in one point and only one point; this common point is called the *point of tangency.* The line joining their centers, or line of centers, passes through the point of tangency. In the case of intersecting circles, the line of centers is the perpendicular bisector of the common chord.

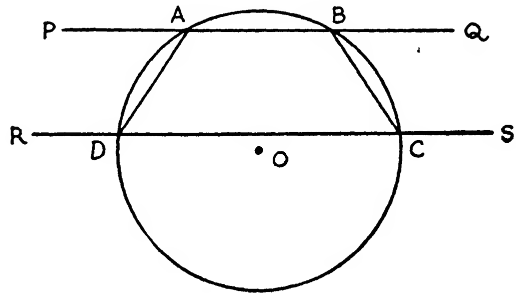


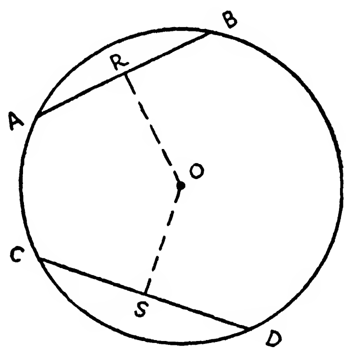

**Chords.** It is readily seen that in the same circle (or in two equal circles), if two chords are equal in length, then their arcs are equal; conversely, if two arcs are equal in length, then their subtended chords are equal. Furthermore, if in the same circle (or in two equal



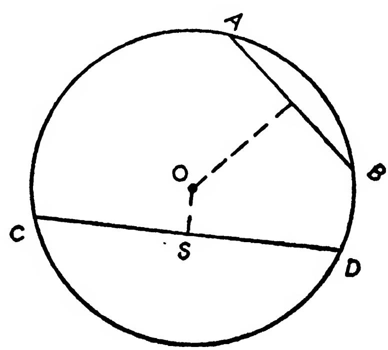
circles) two chords are equal in length, they must be equally distant from the center of the circle; and conversely, if two chords are equally distant from the center, then the chords are equal. If, on the other hand, two chords are *unequal*, then their distances from the center are unequal, and the greater chord is at the smaller distance. In other words, the nearer a chord approaches the center, the longer it becomes, and vice versa. Conversely, if two chords are at unequal distances from the center, the chords are unequal, and the one farthest from the center is the shorter chord. Any line passing through the center of a circle and perpendicular to a chord will bisect both the chord and its subtended arc; conversely, any perpendicular bisector of a chord must pass through the center of the circle.



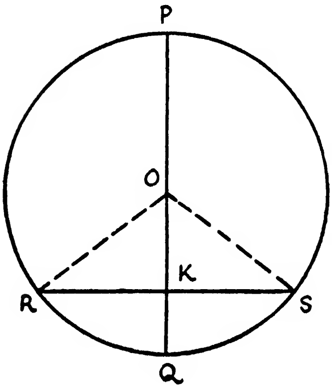



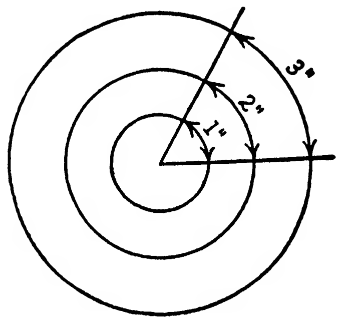

**Circles and Angles.** Any angle formed by two radii is called a *central angle;* its vertex is, of course, at the center. A central angle is said to be measured by its *intercepted arc;* this means that the number of degrees in the arc is the same as the number of degrees in its central angle, and conversely. An arc, in other words, may be described as being of so-and-so many degrees, or as so many inches, etc., long. How to measure an arc in *linear* units instead of *angular* units will be described later; here it should be



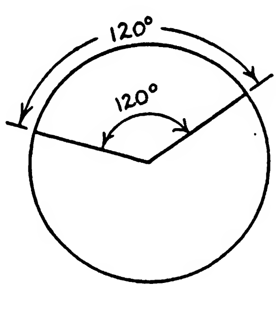
noted that in a series of *concentric circles,* or circles having the same center but different radii, a central angle of a fixed number of degrees will cut off arcs of different *absolute* lengths, even though all of these arcs are of the same *relative* length, i.e., contain the same number of angular units, or degrees. Or, putting it another way, the *actual* length of an arc depends not only upon the number of degrees of arc, but also on the radius of the circle.

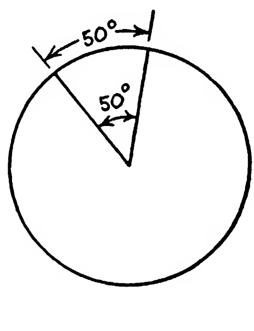
In any given circle, therefore, or in any two equal circles, equal central

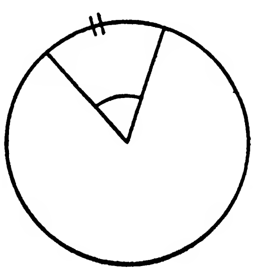


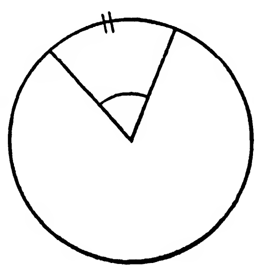




angles intercept equal arcs, and equal arcs are intercepted by equal central angles. Furthermore, if two central angles are unequal, their intercepted arcs are unequal, and the greater angle intercepts the greater arc; conversely, if two arcs are unequal, their central angles are unequal, and the greater arc is intercepted by the greater angle.

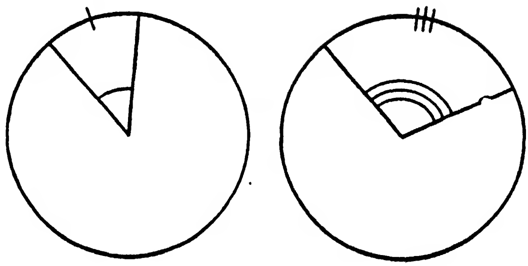


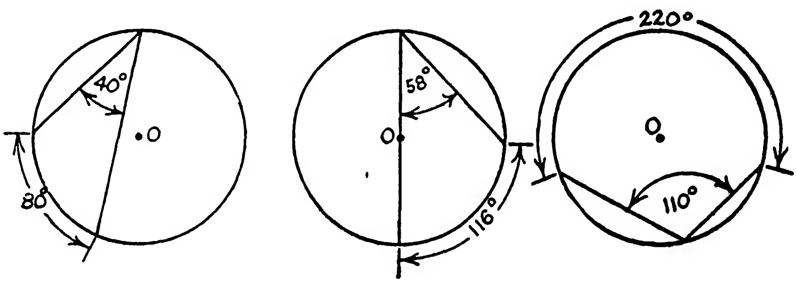



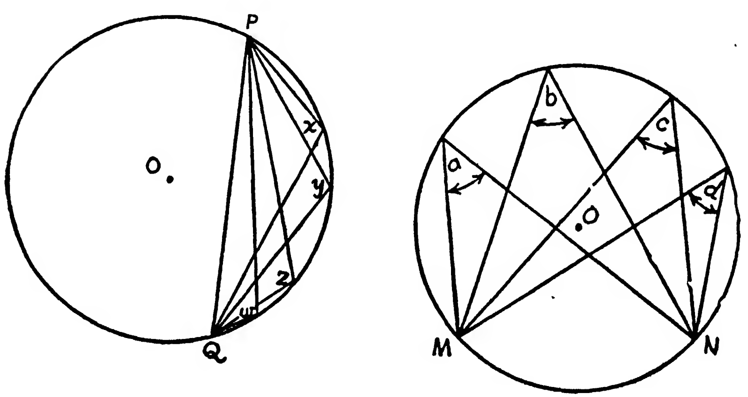


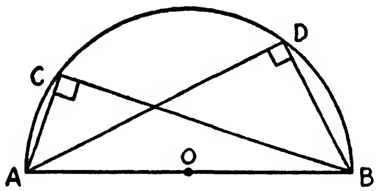
An inscribed angle of a circle refers to

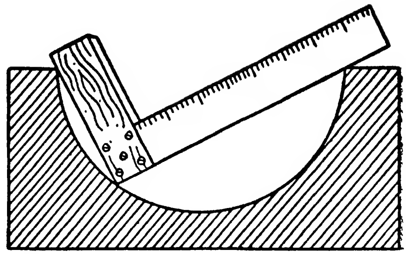


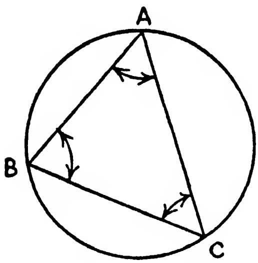
any angle formed by two chords (or a chord and a diameter) which intersect on a point on the circle. An inscribed angle is always measured by *half* its intercepted arc; or, the arc contains *twice* as many degrees as the inscribed angle which intercepts it. From this it will be seen that *all* angles inscribed in the same *segment* are equal, a segment being a figure bounded by an arc and its subtended chord. Thus in the figure, $\angle x = \angle y = \angle z = \angle w$; also, $\angle a = \angle b = \angle c = \angle d$, etc. As a special case of this, we note that all angles inscribed in a semicircle must be right angles. This is the relation a carpenter uses to test the accuracy of a semicircular groove; if the vertex of the carpenter's square touches every point of the groove as the square slides around, the groove is a true semicircle.

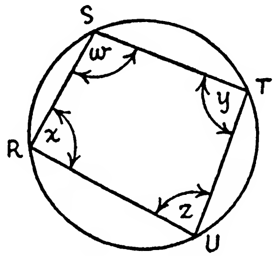


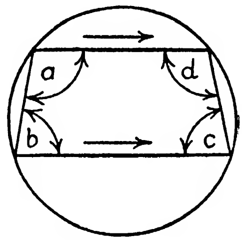








Other interesting and significant relations also become clear.

(1) Verification of the sum of the angles of a triangle;

$$\angle A = \tfrac{1}{2}\ \overset{\frown}{BC}$$
$$\angle B = \tfrac{1}{2}\ \overset{\frown}{AC}$$
$$\angle C = \tfrac{1}{2}\ \overset{\frown}{AC}$$

$$\angle A + \angle B + \angle C = \tfrac{1}{2}(\overset{\frown}{BC} + \overset{\frown}{AC} + \overset{\frown}{AC}) = \tfrac{1}{2}(360^\circ) = 180^\circ.$$

(2) The opposite angles of an inscribed quadrilateral are supplementary;

$$\angle w = \tfrac{1}{2}\ \overset{\frown}{RUT}$$
$$\angle z = \tfrac{1}{2}\ \overset{\frown}{RST}$$

$$\angle w + \angle z = \tfrac{1}{2}\overset{\frown}{RUT} + \tfrac{1}{2}\overset{\frown}{RST} = \tfrac{1}{2}(\overset{\frown}{RUT} + \overset{\frown}{RST}) = 180^\circ.$$

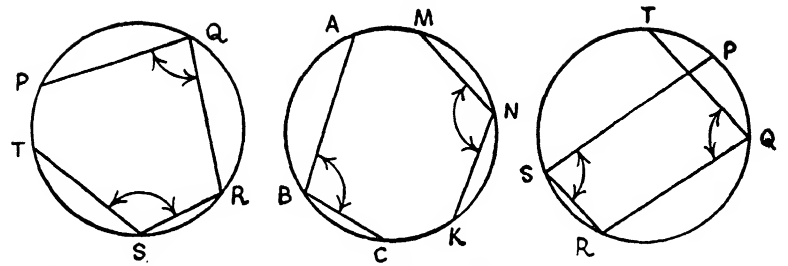
Similarly, $\angle x + \angle y = \tfrac{1}{2}\overset{\frown}{STU} + \tfrac{1}{2}\overset{\frown}{SRU} = \tfrac{1}{2}(360^\circ) = 180^\circ.$

(3) A trapezoid inscribed in a circle must be an isosceles trapezoid;

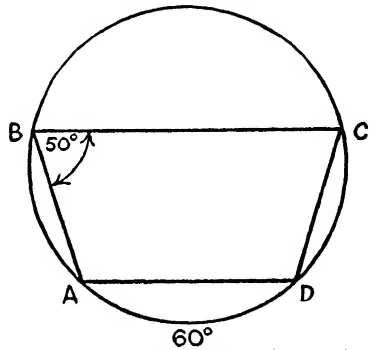
$\angle b + \angle a = 180^\circ$, or $\angle b = 180^\circ - \angle a$
$\angle c + \angle a = 180^\circ$, or $\angle c = 180^\circ - \angle a$
therefore $\angle b = \angle c$.

### *Exercise 59.*

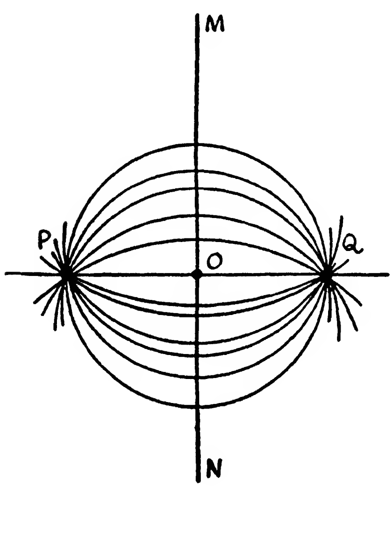
1. **Find the sum of $\angle Q + \angle S$, if arc PT$=40^\circ$.**
2. **How many degrees are there in $\angle B + \angle N$, if arc AM$=30^\circ$ and arc CK$=40^\circ$?**

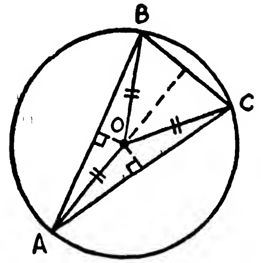



3. Find the sum of $\angle S + \angle Q$, if arc $PT = 44°$.
4. Show why an oblique parallelogram cannot be inscribed in a circle.
5. If ABCD is an inscribed trapezoid, and $\angle ABC = 50°$ and arc $AD = 60°$, find the number of degrees in each of the other arcs.
6. A quadrilateral PQRS is inscribed in a circle, and the two diagonals are drawn; arc $PQ = 80°$, arc $QR = 150°$, and arc $RS = 40°$. Find all the angles in the figure.

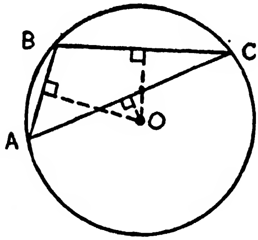



**Circumscribed Circles.** Geometric figures whose vertices lie on a circle and whose sides are chords of the circle, are said to be *inscribed* in the circle; or, the circle is circumscribed about the figure. A little consideration will show that through any *two* given points it is possible to draw an infinite number of circles, each of which has its center somewhere on the perpendicular bisector of the line PQ joining the two given points. But if it is attempted to draw a circle through any *three* given points, only one circle is possible. Thus, one and only one circle may be circumscribed about a given triangle. As we have already seen, the perpendicular bisectors of the three sides of any triangle all meet in a point which is equidistant from the vertices of the triangle. Hence this common point of intersection is the center of the circumscribed circle; its radius is the distance from this point to any one of the vertices ($OA = OB = OC$).

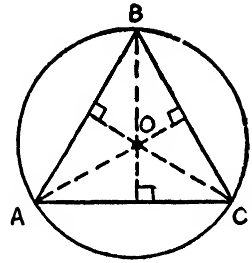




AcuteTriangle

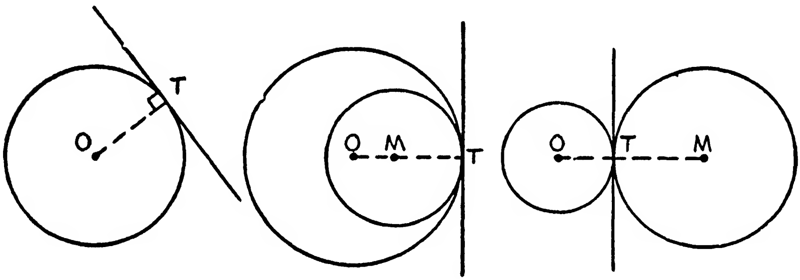


Obtuse Triangle

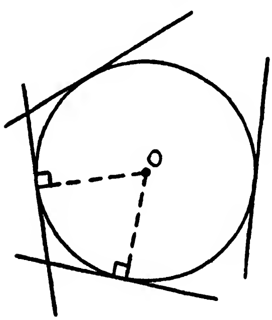


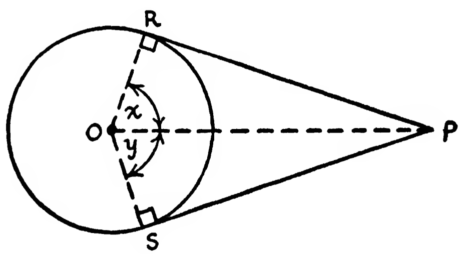
Equilateral Triangle

**Circles and Tangents.** A straight line is said to be *tangent* to a circle if the line touches the circle in one and only one point, no matter how far it is extended. Two circles are said to be tangent if they have but one point in common; they may be *internally* tangent or *externally* tangent. The point of tangency is also called the point of contact. A radius



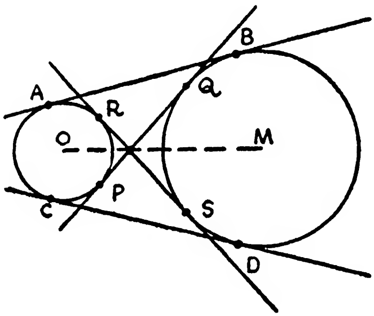

drawn to a tangent at the point of contact is always perpendicular to the tangent; likewise, any line perpendicular to a tangent at the point of contact will pass through the center. In the case of tangent circles, the line of centers passes through the point of tangency.

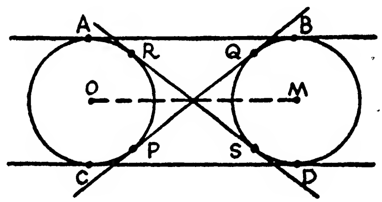


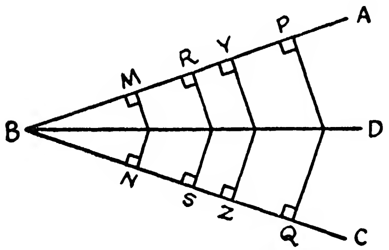



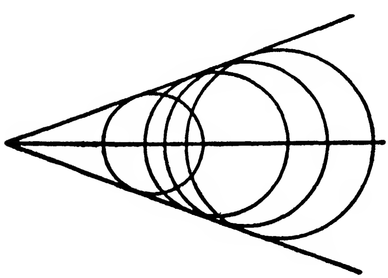

Any number of tangents may be drawn to a given circle, but at different points on the circle; at any *particular* point on a circle *only one*

tangent can be drawn, however. From a point outside a circle, exactly two tangents may be drawn to the circle and no more. These tangents are equal, i. e., PR=PS. Moreover, the line from the outside point (P) to the center bisects the angle between the tangents, i.e., $\angle$ RPO= $\angle$ SPO; also, $\angle x = \angle y$. Where two non-tangent, non-intersecting circles are given, two external and two internal tangents may be drawn. These pairs of tangents are respectively equal, i.e., AB=CD, and RS=PQ. If the circles



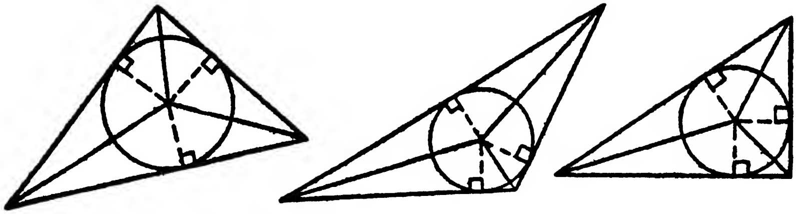



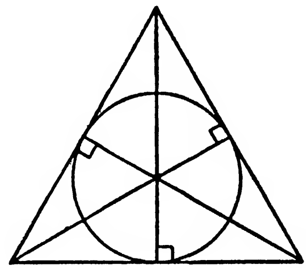
are equal, the external tangents are parallel as well. In both instances the line of centers passes through the point of intersection of the pair of internal tangents.

**The Bisector of an Angle.** Every point on the bisector of an angle lies at equal distances from the sides of the angle. Thus a series of circles could be "inscribed" within the angle, i.e., the sides of the angle would be tangent to the respective circles. This principle is employed when inscribing a circle in a triangle.

**Inscribed Circles.** A circle is said to be inscribed in a figure if it is tangent to every side of the figure. If a circle is inscribed in a triangle, the center of the circle is always the common point of intersection of the three angle-bisectors, and its radius equals the perpendicular distance from the center to the respective sides.

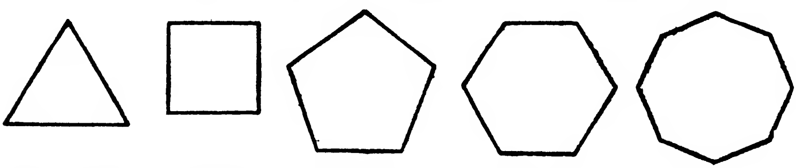


Acute Triangle — Obtuse Triangle — Right Triangle

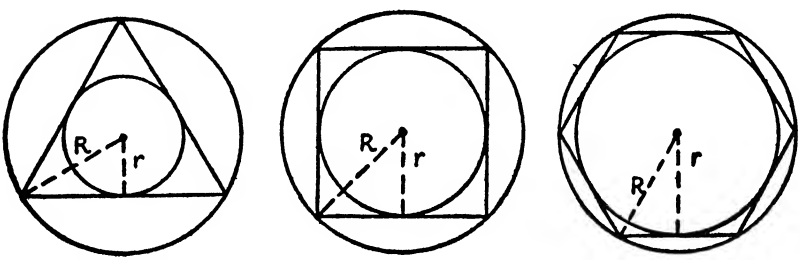


Equilateral Triangle

**Regular Polygons.** If all the sides and all the angles of a polygon are equal, the figure is said to be a *regular polygon.* A polygon may have equal angles, but unequal sides; or, it may have equal sides, but unequal angles; in neither case, however, is it a regular polygon. A circle may always be both inscribed in, and also circumscribed about, any regular polygon, no matter how many sides it may have; the inscribed and circumscribed circles in each case have the same center. The radius of the circumscribed circle is called the radius of the polygon ($R$); the radius of the inscribed circle is called the *apothem* of the polygon ($r$). In the



Equilateral Triangle — Square — Regular Pentagon — Regular Hexagon — Regular Octagon




case of a square, $R$=half the diagonal, and $r$=half the side. In the case of a regular hexagon, $R$ equals the side of the hexagon. In the case of the equilateral triangle. $R=2r$. In all cases, the central angle of a regular

polygon is the angle included between any two consecutive radii; in a regular polygon of $n$ sides, the central angle equals $\left(\frac{360}{n}\right)^{\circ}$.

**Circumference of a Circle.** The length of a circle is called its *circumference.* In every circle, however large or small, the ratio of the circumference ($C$) to its diameter ($D$) is constant, and equal to 3.1416, or approximately $3\frac{1}{7}$ .Thus ratio is called $\pi$ (pi). Thus any circumference is about $3\frac{1}{7}$ times as long as its diameter; that is

$$\frac{C}{D}=\pi=3.1416, \text{ or } C=\pi D.$$

Since $D=2R$, then we also have

$$C=2\pi R.$$

EXAMPLE 1: Find the circumference of a circle with radius 14″.

SOLUTION: $C=2\pi R$

$$C=2\times\frac{22}{7}\times 14$$

$$=2\times\frac{22}{7}\times 14=88'', \textit{ Ans.}$$

EXAMPLE 2: If the circumference of a circle is 110 ft., find its diameter.

SOLUTION: $C=\pi D$, or $D=\frac{C}{\pi}$

$$D=110\div\frac{22}{7}=110\times\frac{7}{22}=35 \text{ ft., } \textit{Ans.}$$

For ordinary rough computations, the value $\pi=3\frac{1}{7}$ may be used; for more exact work, use $\pi=3.14$; and for very accurate work, use $\pi=3.1416$.

**Length of an Arc.** The linear measure of the length of an arc is found by the proportion:

$$\frac{\text{length of arc}}{\text{circumference}}=\frac{\text{central angle}}{360^{\circ}},$$

$$\text{or, arc}=\frac{\text{central angle}}{360^{\circ}}\times 2\pi R.$$

EXAMPLE: Find the length of an arc of 80° in a circle with a diameter of 12″.

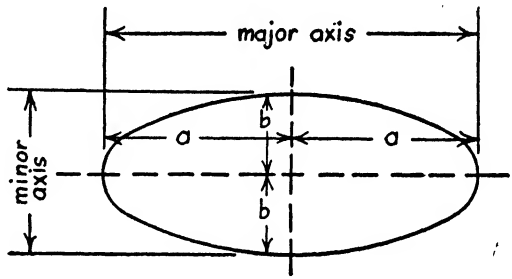
SOLUTION: $$\text{arc}=\frac{80^{\circ}}{360^{\circ}}\times 2\times\frac{22}{7}\times 6,$$

$$\text{or, arc}=\frac{176}{21}=8.38'', \textit{ Ans.}$$

**The Ellipse.** An ellipse is a closed curve with a *major* and a *minor* diameter. It is perfectly symmetrical with diameter, or axis, and is not to be confused with an oval, which is egg-shaped, or narrower at one end than at the other. If the semi-major diameter$=a$, and the semi-minor diameter$=b$, then the length of the perimeter equals

$$P=\pi(a+b)K,$$

$K$ being a constant depending upon the value of $m$, where $m=\dfrac{a-b}{a+b}$. The table shows the value of $K$ for certain values of $m$:




| $m$ | $K$ |
|---|---|
| 0.1 | 1.002 |
| 0.2 | 1.010 |
| 0.3 | 1.023 |
| 0.4 | 1.040 |
| 0.5 | 1.064 |
| 0.6 | 1.092 |
| 0.7 | 1.127 |
| 0.8 | 1.168 |
| 0.9 | 1.216 |

EXAMPLE: Find the perimeter of an ellipse in which $a=5''$ and $b=3''$.

SOLUTION:

$$m=\frac{a-b}{a+b}=\frac{5-3}{5+3}=\frac{2}{8}=.25;$$

hence $K$=midway in value between 1.010 and 1.023, or 1.017 (by interpolation).

Therefore, perimeter$=\pi(a+b)K$

$$=\frac{22}{7}(5+3)\ (1.017)$$

$$=25.6'',\ \textit{Ans.}$$

If $a=b$, then $m=0$ and $K=1$, and the formula becomes $P=\pi(2a)$, or $P=\pi D$, which is what it must be, since if $a=b$, the ellipse has become a circle.

An alternative method for finding the perimeter of an ellipse involves the use of the formula $P=\pi\sqrt{2(a^2+b^2)}$, which also gives an approximate value only, although close enough for most practical purposes. Using this formula for the same problem above, we obtain:

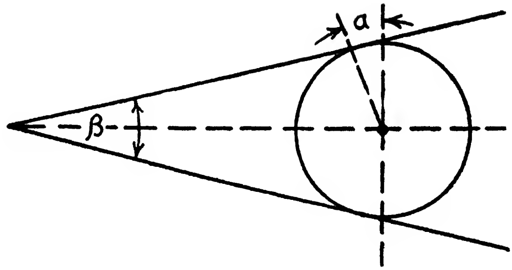
$$P=\pi\sqrt{2(25+9)}=\pi\sqrt{68}$$

$$=\frac{22}{7}\times 8.25=25.9'',\ \textit{Ans.}$$

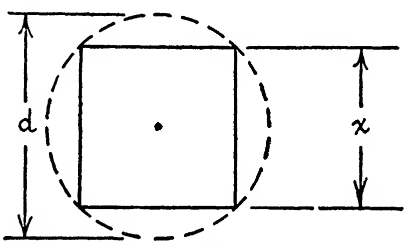
*Exercise 60.*

1. Find the length of a tangent drawn to a circle with a diameter of 8″ from a point 16″ from the center.

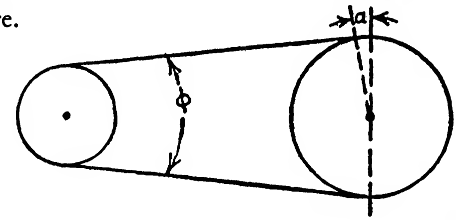
2. Find $\angle a$ if $\angle \beta = 34°20'$; write a formula for $\angle a$ in terms of $\beta$. Assume that the radius to the point of tangency is perpendicular to the tangent.



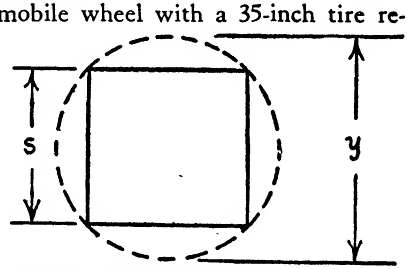

3. Find the side of a square inscribed in a 14-inch circle.
4. What is the value of $x$, if $d = 6.48''$? Write a formula for $x$ in terms of $d$.




5. An electric motor is revolving at 1400 revolutions per minute. Through how many ft. does a point on the rim move in one second if the rim has a diameter of 36″?
6. An automobile wheel with tire has an overall diameter of 3½ feet. When traveling 20 miles per hour, how many revolutions is the wheel making per minute?
7. A square measures 6.4″ on a side. Find the diameter of a circle that will circumscribe this square.

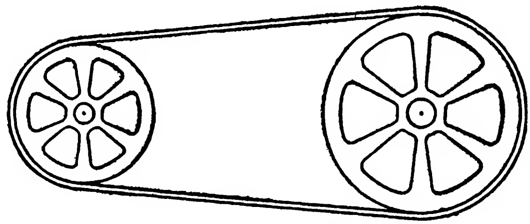



8. If $\angle a = 28°45'$, find $\angle \phi$; write a formula for $\phi$ in terms of $a$.
9. A square 10″ on a side is inscribed in a circle; find (a) the diameter, and (b) the circumference of the circle.
10. How many times must an automobile wheel with a 35-inch tire revolve in going half a mile?

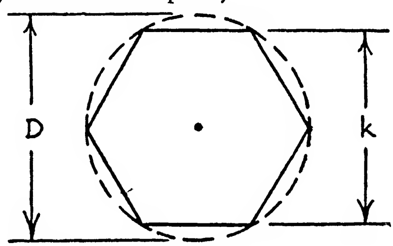



11. Write a formula for finding $y$ when $S$ is known. What is the value of $y$ when $S = .866''$?
12. The rear wheel of a carriage is twice as large in diameter as the front wheel. How much longer is its circumference?
13. A circle 8″ in diameter is circumscribed about a square; find the distance across the flats of the square.

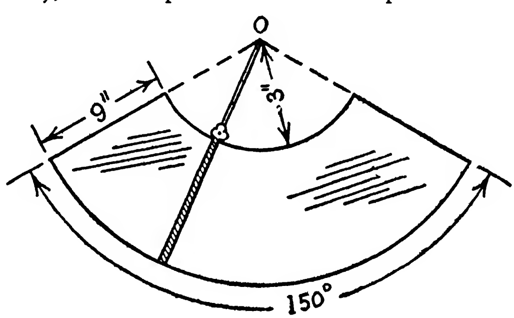
14. An arc of a circle measures 6.6″; if the diameter of the circle is 14″, find the central angle subtended by this arc.
15. A regular hexagon is inscribed in a circle with a diameter of 12″; find the distance across the flats of the hexagon.
16. Two pulleys in a machine shop are connected by a belt. One has a diameter of 18″ and the other a diameter of 12″. For each revolution of the large pulley, how many will the small pulley make?



17. Write a formula for finding $k$ when $D$ is known. What is the value of $k$ when $D=3.12''$?



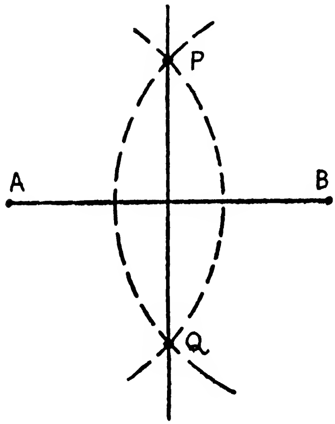

18. Through how many feet does the hub of a wheel travel during one revolution if the radius of the wheel is 8 ft.?
19. A squarehead nut is milled from round stock 1½″ in diameter; find the distance across the flats of the nut.
20. A flywheel 6 ft. in diameter turns at the rate of 120 revolutions per minute. What is the speed per minute of a point on the rim?
21. What diameter of round stock must be used to mill an hexagonal nut 1½″ across the flats?
22. A circular disc has a radius of 4.82″. How many inches in length is an arc on the edge of this disc if its subtended angle contains 156°?
23. A metal die is in the shape of an ellipse. If the major and minor axes are 6″ and 5″, respectively, find the perimeter of the ellipse.
24. How much further will one end of the windshield wiper travel than the other in swinging from left to right?

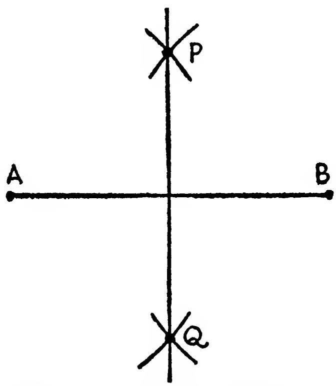



25. A thin cable is wrapped 14 times around the drum of a winch. If the diameter of the drum is 2½ ft., how many feet of cable are wound around the drum?

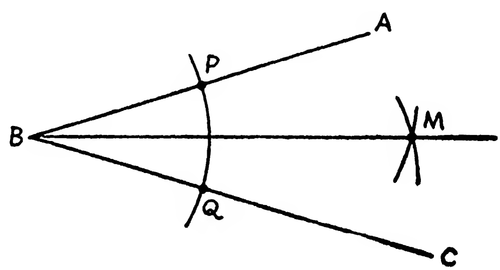
## 16. GEOMETRIC CONSTRUCTIONS

**Bisecting a Given Line.** By the term *geometric construction* is meant a drawing or layout made by using only a pair of compasses and a straight edge. A line may be bisected by swinging equal arcs from each end, and then connecting the points of intersection of the arcs by a straight line. Thus the ends of the line AB are used as centers to draw two arcs which intersect at P and Q; the line then drawn through points P and Q will bisect the line AB; it will also be perpendicular to it. Note that the radius used to swing the arcs must be greater than half the length of AB (why?); also, that it is not necessary to draw the arcs in full, but just enough to locate the points of intersection.

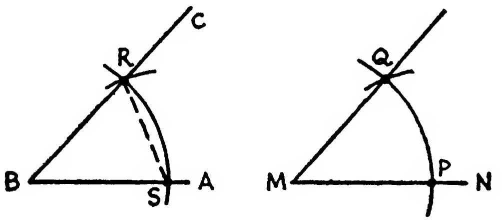


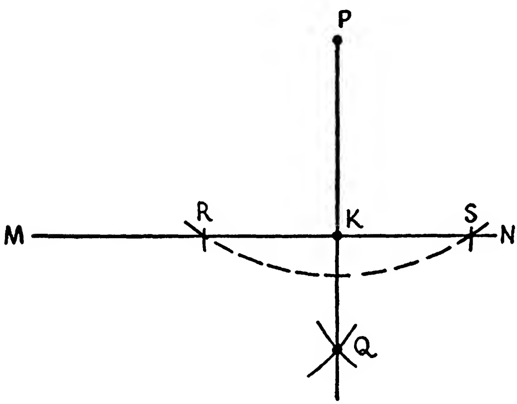

**Bisecting a Given Angle.** If it is desired to divide a given angle ABC into two equal parts, the vertex of the angle, B, is used as a center, and an arc is swung intersecting the sides of the angle in P and Q. Then using the points P and Q as centers, with any convenient radius, swing two equal arcs which intersect at M; draw a straight line connecting B with M. This line bisects $\angle ABC$.



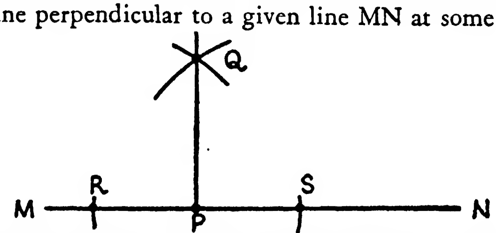
**Constructing an Angle Equal to a Given Angle.** If it is desired to transfer a given angle, i.e., construct an angle equal to a given angle, the procedure is as follows. Let ABC be the given angle which is to be transferred to the line MN. On $\angle$ABC, with B as a center, strike any convenient arc, intersecting the sides in R and S. With the same radius, and a center at M, strike another arc, intersecting MN in P. Then measure the *chord* RS; with this distance RS as a radius, and with P as center, strike another arc intersecting the larger arc in Q. Join M with Q. The angle NMQ equals the given angle ABC.




**Erecting a Perpendicular from a Given Point outside a Given Line.** If from the given point P it is required to construct a line perpendicular to the given line MN, take P as a center and a radius long enough to intersect MN in R and S. Then, with R and S as centers, respectively, strike two arcs with any convenient radius, intersecting in Q. Join P with Q. The line PQ is the perpendicular bisector of the segment RS (why?); hence PK is perpendicular to the given line MN, and, of course, passes through P, as required. (Note that PK does not necessarily bisect MN; also, that MN does not necessarily bisect PQ.)

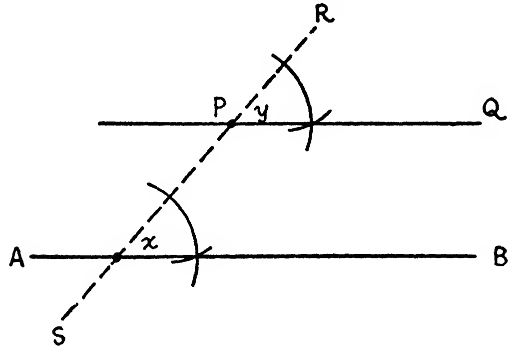



**Erecting a Perpendicular at a Given Point on a Given Line.** Should it be required to construct a line perpendicular to a given line MN at some particular point P *on* the line MN, the procedure is virtually the same. With P as center, and any convenient radius, strike two arcs intersecting MN in R and S. Then, with R and S as centers, and any convenient radius, strike arcs intersecting in Q;

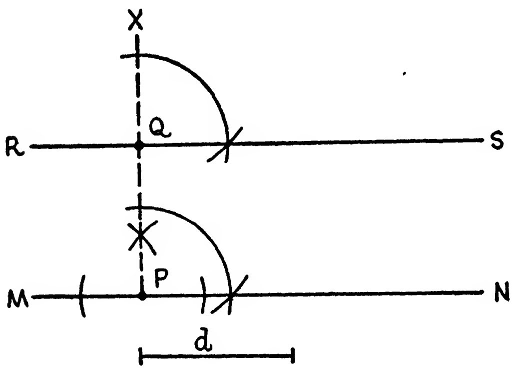


join P with Q. The line PQ is the required perpendicular. If the given point should happen to be at, or very near, either end of the given line, it would first be necessary to extend the line somewhat; then proceed as before.

**Constructing a Line Parallel to a Given Line through a Given Point.** It is desired to draw a line through P, and parallel to the line AB. Draw RS through P at any convenient angle to AB. Then construct $\angle y = \angle x$, using PR as one side of $\angle y$. The line PQ will be parallel to AB, and, of course, passes through P, as required.

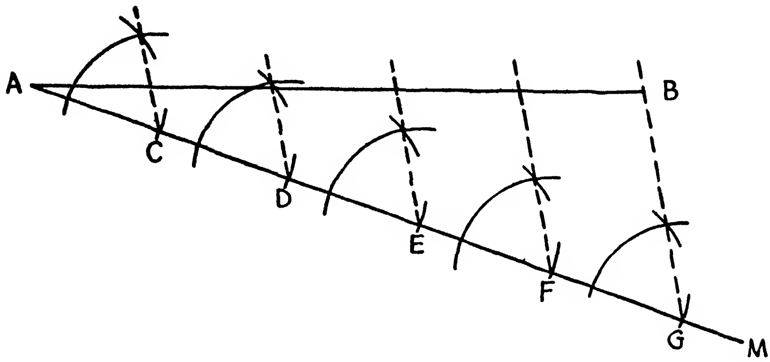



**Constructing a Line Parallel to a Given Line at a Given Distance from the Line.** It is desired to construct a line which will be parallel to the given line MN, and also at a given distance, *d*, from MN. The procedure follows. At any convenient point P on MN, erect a perpendicular PX, as described above. Then on PX, beginning at P, measure off a length PQ equal to the given distance *d*. Now through Q construct the line RS parallel to MN (by constructing a right angle at Q, using QX as one side). The line RS is the required line, for it is not only parallel to MN, but is also at the given distance *d* from MN.



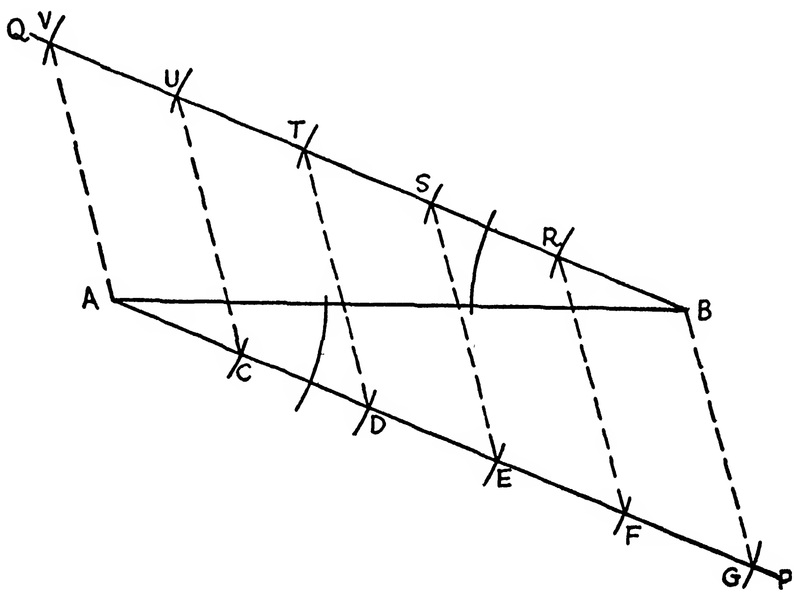

**Dividing a Given Line into Any Number of Equal Parts.** If it is desired to divide a line of given length into any number of equal parts, either of the following methods may be used.

(a) Suppose the line AB is to be divided into five equal segments; draw line AM at any angle to AB, and long enough to lay out the required number of divisions conveniently. Step off five equal segments on line AM,




starting at A; connect the last point (G) with point B. Then draw lines through C, D, E and F parallel to BG. The intersections of these lines with AB will divide it into the required number of equal parts.

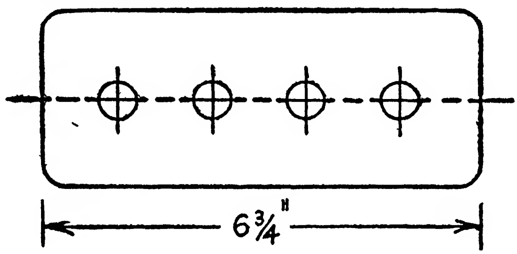
(b) The alternative method consists of drawing two lines, AP and BQ. parallel to each other, through A and B respectively, at any convenient



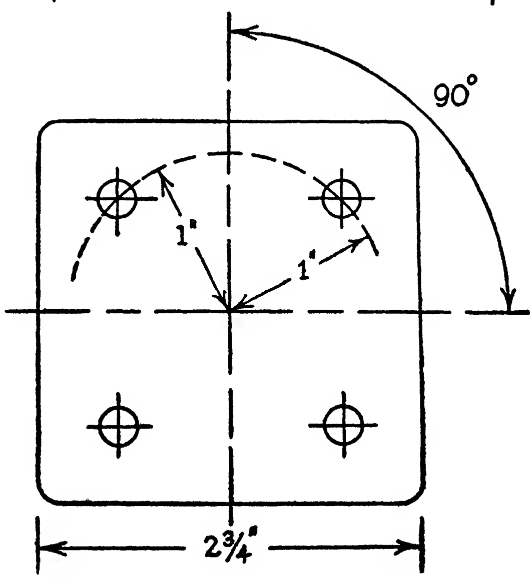

angle to AB. Then lay off the same number of equal segments on AP and BQ, starting in each case from A and B, respectively. Connect AV, CU, DT, ES, FR, and GB. The intersections of these lines with the line AB will divide it into the required number of equal parts.

*Exercise 61.*

1. Draw a horizontal line 4⅝″ long, and bisect it geometrically; do the same for a vertical line 5$\frac{9}{16}$″ long.
2. Lay out an angle of 70° with a protractor; then bisect the angle, showing all construction lines.
3. With a protractor, draw an angle of 110°; bisect this angle geometrically.
4. Lay out an angle of 27° with a protractor and bisect this angle.
5. Erect a perpendicular to a line 3¾″ long at a point 1½″ from either end of the line.
6. Erect a perpendicular to a line 3⅞″ long at either end of the line. (*Hint:* Extend the line first.)
7. Draw an oblique line 4¼″ long; then mark a point, anywhere, approximately 2″ from either side of the line. Now construct a perpendicular to the line from that point.
8. Draw a line 5⅝″ long and divide it into 7 equal parts.
9. Four equally spaced holes are to be drilled in the plate shown; the end holes are to be as far from the edge as the distance between the holes. Locate the centers by means of a geometric construction.

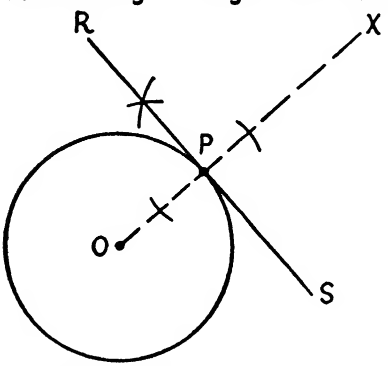



10. Construct an angle of 90°. Then construct an angle of 45° by bisecting the right angle. Construct an angle of 22½° by bisecting the 45° angle.
11. Lay out (with the protractor) an angle of 60°. Now construct, by bisection, an angle of 30°; of 15°; of 7½°.



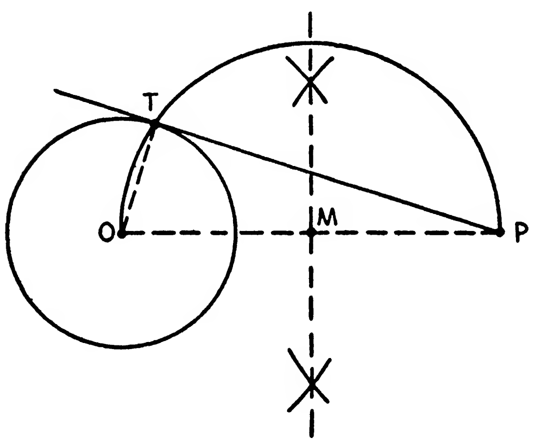
12. Draw an oblique line. Construct a line parallel to this line and at a distance of 1⅞″ from it.
13. Construct a square, 2¾″ on a side; locate its center. Then locate the centers of the four holes as indicated.
14. Draw any acute triangle. Construct (a) an altitude to the longest side; (b) a medium to the shortest side.
15. Construct a right triangle whose short sides are 1½″ and 2½″. Construct (a) the perpendicular bisectors of all three sides; (b) the altitude upon the hypotenuse.

**Constructing a Tangent to a Circle at a Given Point on the Circle.**

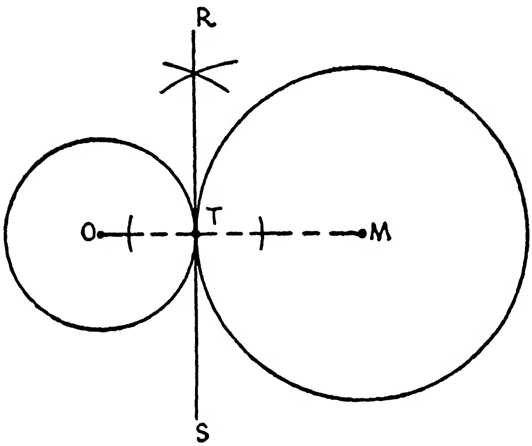



To the given circle O, it is required to construct a tangent touching the circle at the given point P. *Construction:* Draw the radius OP, and extend OP a convenient distance beyond the circle to X. Construct the line RS perpendicular to OX at P. The required tangent to circle O at point P is the line RS.

**Constructing a Tangent to a Given Circle from a Given Point outside the Circle.** *Construction:* Join the center O of the given circle with the given point P. Determine the midpoint M of OP by constructing the perpendicular bisector of OP. With M as center and MO as a radius, construct a semicircle on OP as diameter, intersecting the given circle in point T. Join T with P, and extend PT, which is the required tangent to the circle from the given point P. (why?) (*Hint:* What can you say of ∠OTP?) The other tangent to the circle from P could of course be constructed in the same way by completing the circle on OP as diameter.

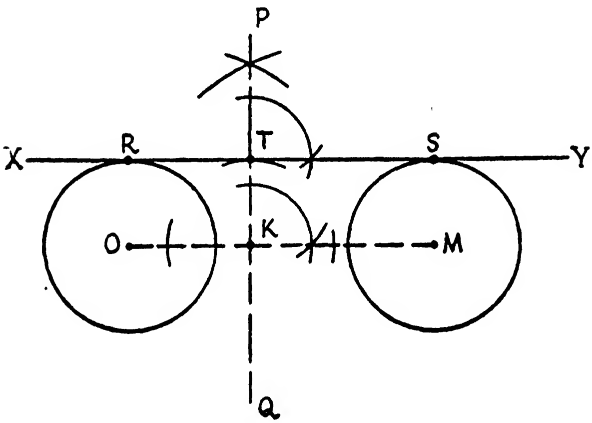


**Constructing a Common Tangent to Two Externally Tangent Circles.**

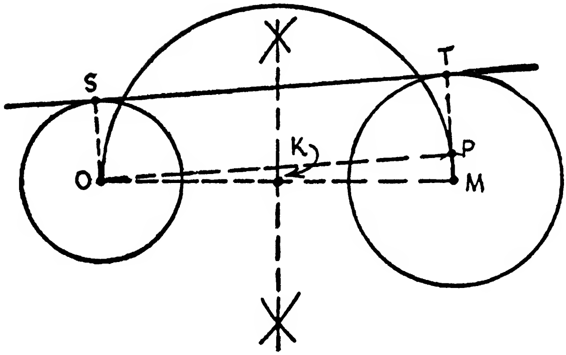



*Construction:* Join the centers O and M of the two given tangent circles. This line of centers will pass through the point of tangency, T. At T, construct a perpendicular to OM. The line RS is the required common tangent to the two circles.

**Constructing a Common External Tangent to Two Given Equal Circles.** *Construction:* Draw the line of centers, OM. At any convenient point on OM, say K, erect a perpendicular to OM, say PQ. On PQ, beginning at K, lay off a distance KT equal to the radius of either of the two equal circles. Through T construct a line XY parallel to OM (or perpendicular to PQ). The line XY will be tangent to the circles at R and S respectively, as required.

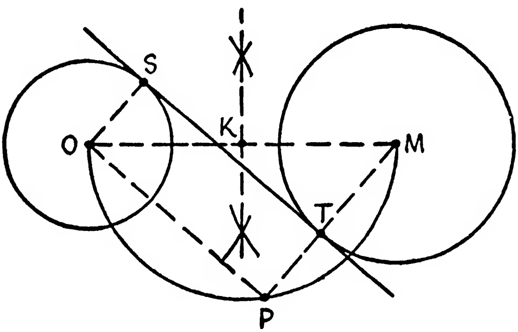



**Constructing a Common External Tangent to Two Given Unequal Circles.** *Construction:* Draw the line of centers OM; bisect OM in K.




With K as center and OK as radius, construct a semicircle on OM as diameter. Then with M as center, and a radius equal to the *difference between the lengths of the radii of the given circles* $(R—r)$, strike an arc intersecting the semicircle in P. Join M with P and extend to T, the intersection with the circle. Draw OP. Now construct a line through T parallel to OP and extend it; it will touch the other circle at S, and ST is the required external tangent. An alternative procedure would be, instead of constructing TS parallel to OP, to erect a perpendicular to OP at O, and extend this perpendicular to the circle at S; join S with T.

**Constructing a Common Internal Tangent to Two Given Unequal Circles.** *Construction:* Draw the line of centers OM; bisect OM in K.




With K as center and OK as radius, construct a semicircle on OM as diameter. Then with M as center, and a radius equal to *the sum of the lengths of the radii of the given circles* $(R+r)$, strike an arc intersecting the semicircle in P. Join M with P, and let MP intersect the given circle in T. Draw OP. Now construct a line through T parallel to OP and extend it; it will touch the other circle in S, and ST is the required internal tangent.

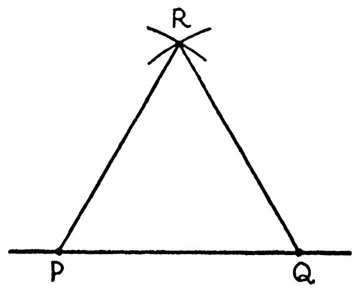
An alternative procedure would be, instead of constructing TS parallel to OP, to erect a perpendicular to OP at O, and extend this perpendicular to the circle at S; join S with T.

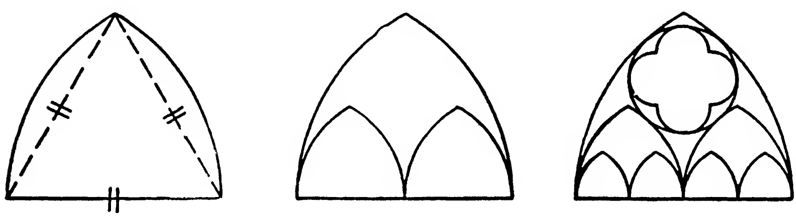
### *Exercise 62.*

1. Draw a circle having a radius of 2½″; select any point on the circle and construct a tangent to the circle at that point.

2. Draw a circle with a radius of 1⅞″ and extend any radius to a point P such that P is 3¼″ from the center. Construct a tangent to the circle from P.
3. Draw two externally tangent circles, one with a radius of 1½″, the other with a radius of 2¼″.
4. Construct two circles, each with a radius of ¾″, having their centers 3″ apart. Construct a common external tangent to the two circles.
5. Show by construction how two pulleys each with a 3″ diameter are connected (a) by an open belt; (b) by a crossed belt.
6. Show by construction how a 2″ pulley and a 3½″ pulley are connected by (a) an open belt; (b) a crossed belt.
7. Draw any obtuse triangle. Construct (a) the perpendicular bisectors of the three sides; (b) the circumscribed circle of the triangle.

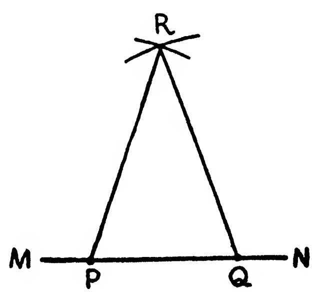
**Constructing an Equilateral Triangle.** If it is desired to construct an equilateral triangle with a given length as its side, all that needs be done is to lay off a segment PQ equal to the required side; then, with P and Q as centers, respectively, and with PQ as a radius, strike off two arcs intersecting in R. Join R with P and with Q; the triangle PQR is the required equilateral triangle. This construction is the basis of the well-known Gothic arch of medieval times, commonly used in cathedrals and other public buildings.



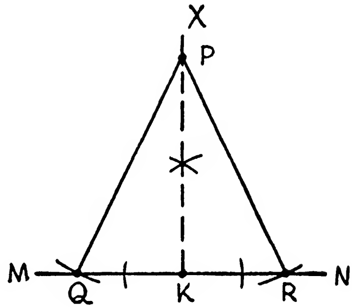



**Constructing an Isosceles Triangle.** Several possibilities may arise.

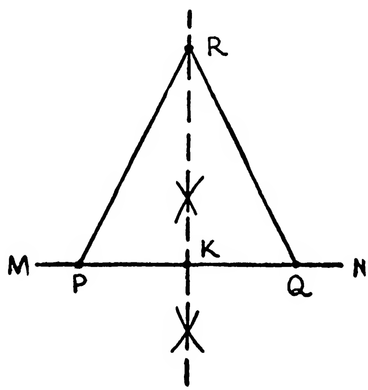
(1) *Given the base and the length of each of the two equal sides.* On any line MN lay off the required base, PQ. With P and Q as centers, respectively, and a radius equal to the given equal sides, strike two arcs intersecting in R. Triangle PQR is the required isosceles triangle.



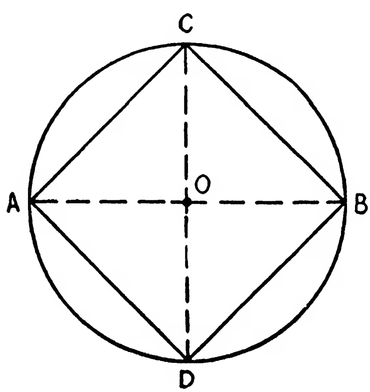
(2) *Given the altitude and the two equal sides.* At any convenient point K on any line MN, erect a perpendicular to MN and extend it to X. On KX, beginning at K, lay off a distance KP equal to the required altitude. With P as a center and a radius equal to the length of the required equal sides, strike two arcs intersecting MN in Q and R. Triangle PQR is the required isosceles triangle.



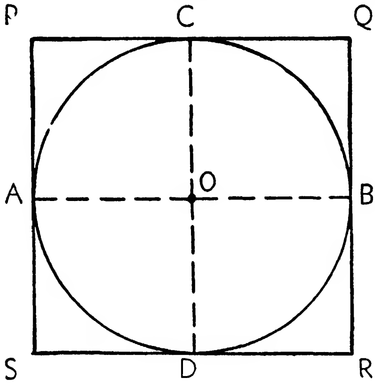

(3) *Given the base and the altitude.* On any line MN lay off the required base, PQ. Bisect PQ in K; at K erect a perpendicular to MN. Beginning at K, lay off a distance KR equal to the required altitude. Join R with P and with Q. Triangle PQR is the required isosceles triangle.



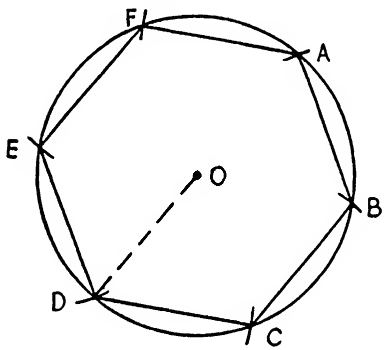

**Inscribing a Square in a Given Circle.** Draw any convenient diameter AB; then construct another diameter, CD, perpendicular to AB. Join points A, C, B, and D; the resulting figure is the required inscribed square.



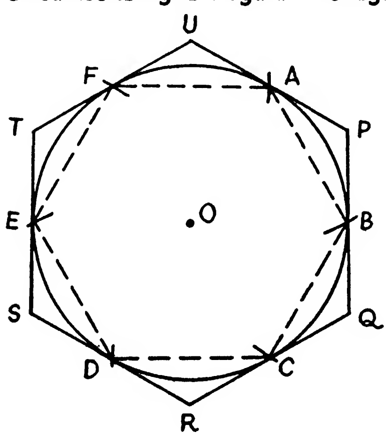
**Circumscribing a Square about a Given Circle.** Construct two perpendicular diameters, AB and CD. At each of the four extremities A, B, C, and D construct a tangent to the circle. The figure PQRS is the required circumscribed square.



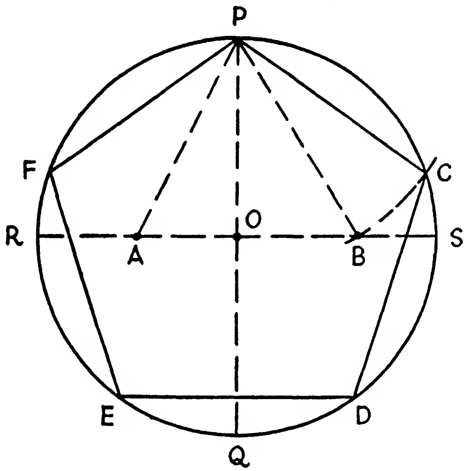

**Inscribing a Regular Hexagon in a Given Circle.** Starting with any point A on the given circle, and using a radius equal to the radius OA of the given circle, strike off successive arcs at B, C, D, E and F. Join the six points A, B, C, D, E and F; the resulting figure is the required inscribed hexagon. Each side of this hexagon is equal in length to the radius of the circle in which it is inscribed.




**Circumscribing a Regular Hexagon about a Given Circle.** Divide the given circle, as above, into six equal arcs, the points of division being A, B, C, D, E and F. At each of these points of division construct a tangent to the circle. The figure PQRSTU is the required circumscribed hexagon.



**Inscribing a Regular Pentagon in a Given Circle.** Construct any two perpendicular diameters, such as RS and PQ. Now bisect RO, and join the mid-point A with point P. With a radius equal to PA, lay off AB=PA; join P and B. Then with a radius equal to PB, and with P as center, strike an arc BC, cutting the circle in C. The chord PC is the side of the required pentagon; step off arc PC five times, beginning at P, and the figure PCDEF will be the desired regular inscribed pentagon.



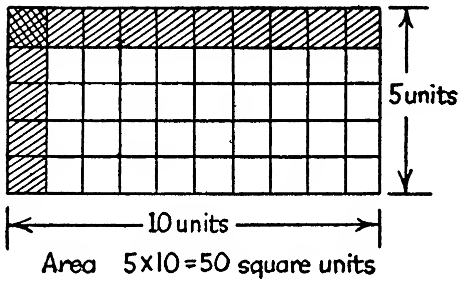

*Exercise 63.*

1. Construct an equilateral triangle 1½″ on a side.
2. Construct an angle of 60°; by bisection, construct an angle of 30°; 15°.
3. Show how to construct (with a protractor) an angle of 120°; 75°; 105°; 67½°; 135°; 150°; 165°.
4. Construct an isosceles triangle having a base 2½″ and its equal sides each 3½″ long.
5. Construct an isosceles triangle with a base of 3″ and an altitude of 1¼″.
6. Construct an isosceles triangle with an altitude of ¾″ and equal sides of 1½″ each.
7. Construct a Gothic arch on a base line of 1½″.
8. Construct an equilateral triangle 1⅜″ on a side; then construct the inscribed and circumscribed circles.
9. Construct an isosceles triangle with a base 2¼″ long and base angles of 30° each.
10. Construct an isosceles triangle with equal sides of 1″ each and a vertex angle of 120°.
11. Construct a 30°—60°—90° triangle with an hypotenuse of 2⅝″.
12. Construct an isosceles right triangle 1¼″ on each of the two equal sides.
13. Construct the largest possible square in a 1¾″ circle.
14. Show by construction the largest square which can be milled from a piece of 1½″ round stock.
15. Show by construction the largest hexagon that can be milled on a 2″ round shaft.

16. Draw a circle with a radius of $1\frac{1}{2}''$. Inscribe a rectangle in this circle, having a width of $1''$.
17. Construct a $30°—60°—90°$ triangle such that its shortest side is $1\frac{1}{2}''$ long; now construct its inscribed and circumscribed circles.

## 17. MEASUREMENT OF AREAS

**Area of Squares and Rectangles.** Surface measure, or the measure of area, is expressed in square units, such as the square inch, the square foot, or the square centimeter. Thus the area of a plane geometric figure may be described as the number of units of surface measure contained in the figure in question. A square inch is defined as a square, each of whose sides measures one linear inch; similarly for a square foot, etc. It is readily seen that the area of a rectangle may be found simply by multiplying the length of the rectangle by its width; or, expressed as a formula:

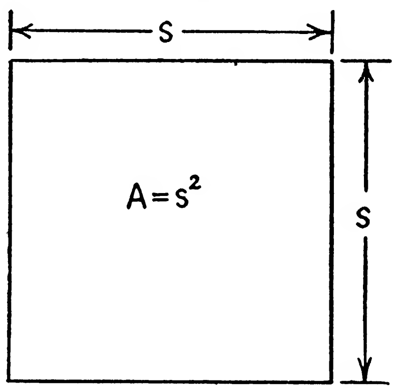



$$A=lw.$$

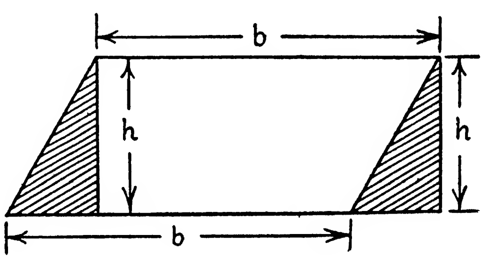
For a square, this becomes

$$A=s\times s,$$

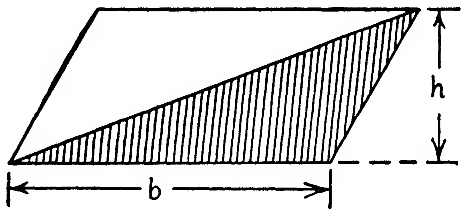
or $A=s^2$.




**Area of a Parallelogram.** In the case of a parallelogram, the area is found by multiplying the base by the altitude, since from the diagram it is clear that the shaded triangles are equivalent in area; also, the parallelogram is equivalent in area to the rectangle, which has the same base and altitude, respectively, as the parallelogram. Hence the area of a parallelogram is $A=bh$.



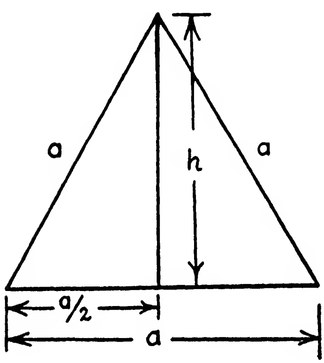
**Area of a Triangle.** When it is remembered that a parallelogram is divided into two equivalent triangles by either of its diagonals, one method of computing the area of a triangle is readily suggested. The base of the shaded triangle is equal to the base of the parallelogram, and the altitude of this triangle is equal to that of the parallelogram; therefore, since the area of the triangle is half that of the parallelogram, the area of the triangle is half of the base times the altitude; or, $A=\frac{1}{2}bh$.




To use this formula it is necessary to know the length of the altitude drawn to one of the three sides, as well as the length of that side. This may not always be known or conveniently measured. If, for example, we know only the lengths of each of the three sides, the following formula may be used instead:

$A=\sqrt{(s)\ (s-a)\ (s-b)\ (s-c)}$, where $a$, $b$, and $c$ represent the lengths of the three sides, respectively, and $s$ equals the *semiperimeter,* or $\frac{1}{2}(a+b+c)$. Thus, if the sides of a triangular sheet of metal measure 8″, 12″, and 16″, the area of the sheet equals $A=\sqrt{(18)\ (18-8)\ (18-12)\ (18-16)}=\sqrt{(18)\ (10)\ (6)\ (2)}=\sqrt{2160}=$ 46.4 sq. in. As a special case, the area of an equilateral triangle yields another formula; since in this case $a=b=c$, the above formula becomes:

$$A=\sqrt{\left(\frac{3a}{2}\right)\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)}=\frac{a^2}{4}\sqrt{3},$$




where $a$ represents any one of the three equal sides. The same result can also be obtained by using the altitude and base directly, as follows:

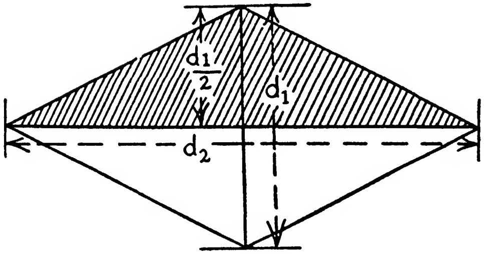
$$h=\frac{a}{2}\sqrt{3};$$

but $A=\frac{1}{2}(\text{base})\times(\text{altitude})$; therefore

$$A=\frac{1}{2}(a)\left(\frac{a}{2}\sqrt{3}\right),\ \text{or}\ A=\frac{a^2}{4}\sqrt{3},$$

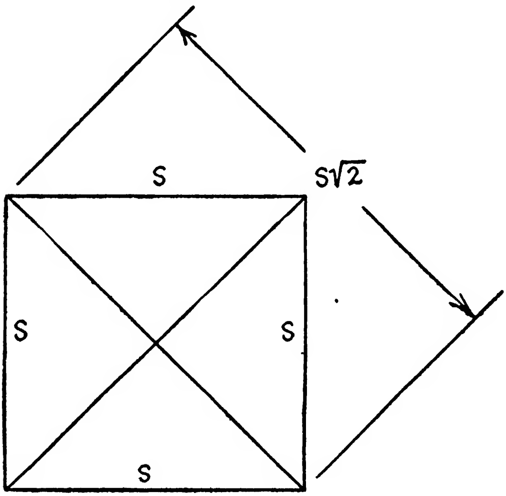
as before. Thus, if the side of an equilateral triangle is 6″, its area equals $\frac{36}{4}\times\sqrt{3}=9\times1.73=15.6$ sq. in.

**Area of a Rhombus in Terms of the Diagonals.** Since the diagonals of a rhombus not only bisect each other, as in all parallelograms, but inter-

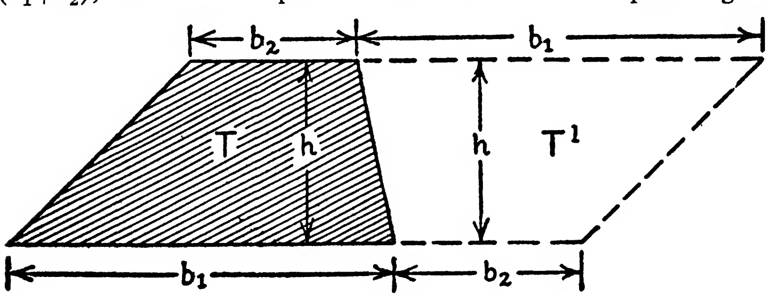



sect at right angles as well (which is true only in a rhombus or a square, i.e., an equilateral parallelogram), then the base of the shaded triangle $=d_2$ and its altitude$=\frac{d_1}{2}$; hence the area of the shaded triangle is ½ $(d_2)\left(\frac{d_1}{2}\right)$, or ¼ $d_1d_2$. Therefore the area of the entire rhombus (being twice as large as the shaded triangle) is given by: $A=2(\frac{1}{4}d_1d_2)$, or $A=\frac{1}{2}d_1d_2$.

An interesting illustration of this relation is found by applying it to a square, where each diagonal equals the other, and is given (as we have already seen) by $d=s\sqrt{2}$, where $s$ represents the side of the square. Applying the above formula, then, we obtain: $A$ (of a square)$=\frac{1}{2}(s\sqrt{2})\times(s\sqrt{2})$, or $A=s^2$, as before.



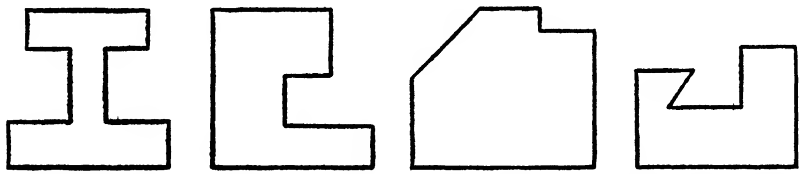

**Area of a Trapezoid.** The area of the entire parallelogram is evidently $h(b_1+b_2)$; since either trapezoid is half the area of this parallelogram,



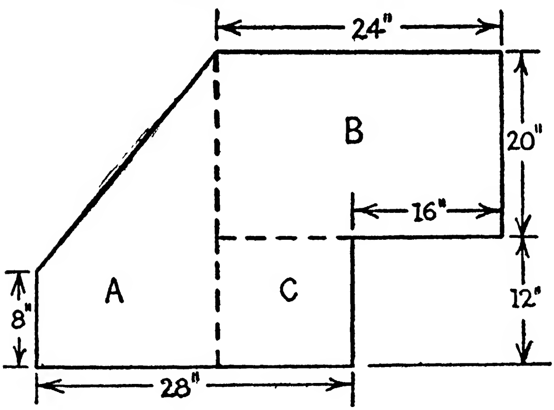
then the area of the trapezoid must be given by $A=\frac{1}{2}h(b_1+b_2)$. Since the median of a trapezoid equals one half the sum of its bases, the area of a trapezoid is also equal to the altitude times the median; or,

$$A=hm, \text{ where } m=\tfrac{1}{2}(b_1+b_2).$$

**Irregular Structural Shapes.** Metal plates and other structural parts are often in the shape of irregular polygons as here shown. If there are no



rounded corners or other curved lines in the figure (which is frequently the case, however) the area may be computed by decomposing the figure into convenient rectangles, triangles, trapezoids, etc., taking the appropriate measurements, finding the area of each part, and then simply adding them together.

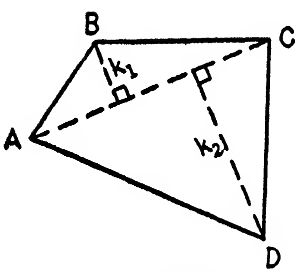



EXAMPLE: Find the area of the iron plate shown, with dimensions as given.

SOLUTION: Break up the original figure into trapezoid A and rectangles B and C.

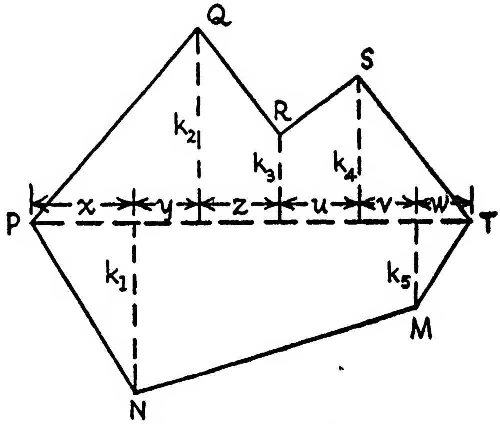
| | | | |
|---|---|---|---|
| Trapezoid | $A=\frac{1}{2}h(b_1+b_2)=\frac{1}{2}(20)(8+32)$ | | $=400$ |
| Rectangle | $B=$ | $24\times20$ | $=480$ |
| Rectangle | $C=$ | $12\times 8$ | $=\ 96$ |
| | Total figure | | $=976$ sq. in., *Ans.* |

A somewhat more general method, particularly for irregularly shaped figures, consists of drawing a convenient diagonal, dropping and measuring the perpendiculars to this diagonal from each remaining vertex, and measuring the projections of these perpendiculars upon the diagonal as well; with this data it is a simple matter to find the area of each of the separate triangles and trapezoids into which the figure has been decomposed, as suggested below.




Area of ABCD

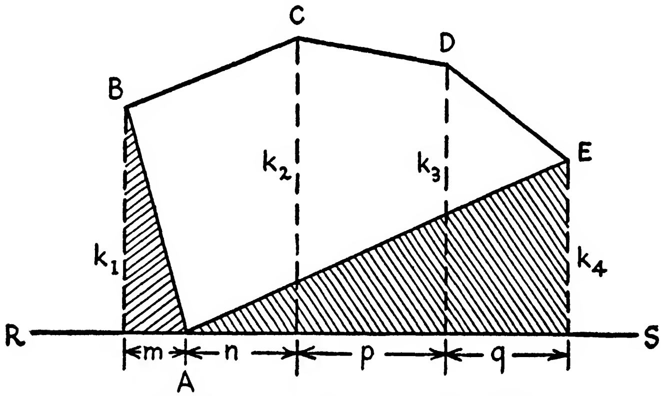
$=\frac{1}{2}(AC)k_1+\frac{1}{2}(AC)k_2$
$=\frac{1}{2}(AC)(k_1+k_2)$




Area of MNPQRST

$=\frac{1}{2}(x+y)k_2+\frac{1}{2}(z)(k_2+k_3)+$
$\frac{1}{2}(u)(k_3+k_4)+$etc.

Usually the modified method of using a base line (RS) instead of the diagonal is slightly more convenient. Thus the area of ABCDE is given by the sum of the areas of the three trapezoids diminished by the sum of the areas of the two triangles; or,

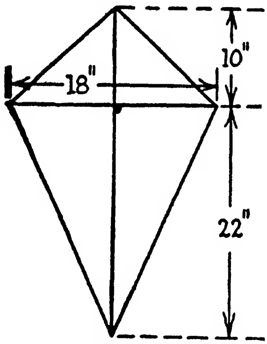


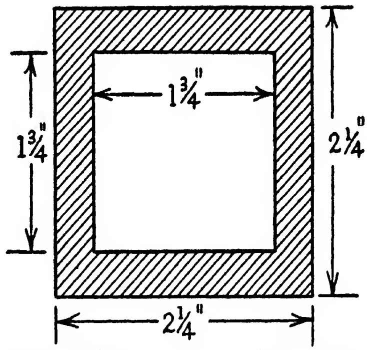

$$A=\frac{1}{2}(m+n)(k_1+k_2)+\frac{1}{2}(p)(k_2+k_3)$$
$$+\frac{1}{2}(q)(k_3+k_4)-\frac{1}{2}mk_1$$
$$-\frac{1}{2}(n+p+q)(k_4).$$

*Exercise 64.*

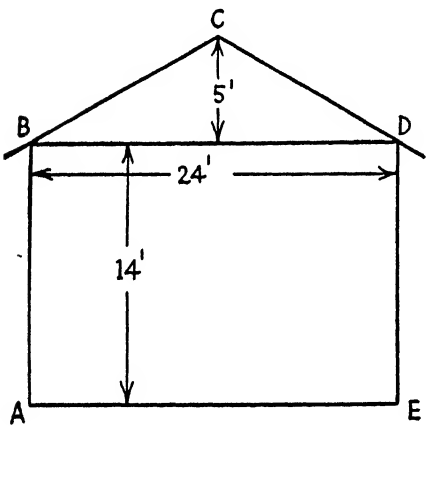
1. Find the area of a 30°—60°—90° triangle whose hypotenuse is 14″.
2. Find the area of an isosceles triangle whose base is 6″ and whose equal sides are each 8″ long.

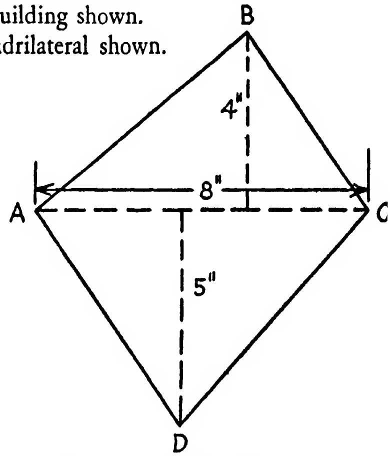
3. The sides of a parallelogram are 6″ and 12″, and its diagonal is 16″. What is its area?



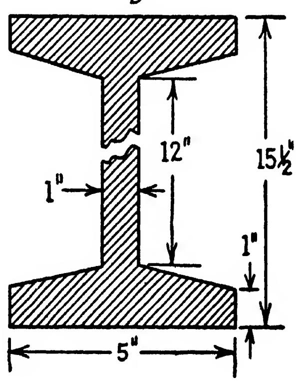




4. How many square inches of paper are needed to cover this kite frame?
5. What is the area of a fiber gasket cut in the dimensions shown?
6. Find the area of the front of the building shown.
7. Find the area of the irregular quadrilateral shown.

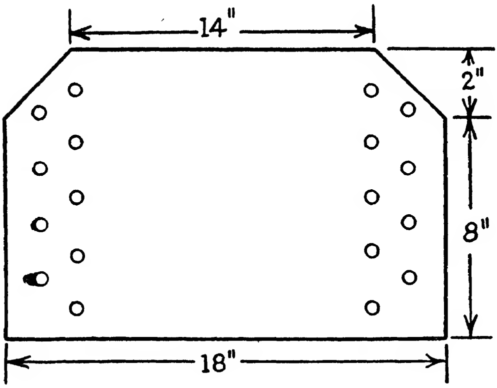


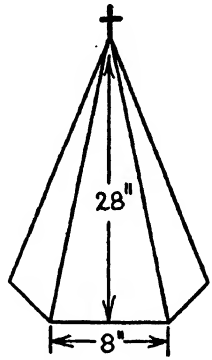




8. What is the cross-sectional area of the beam shown in the sketch?

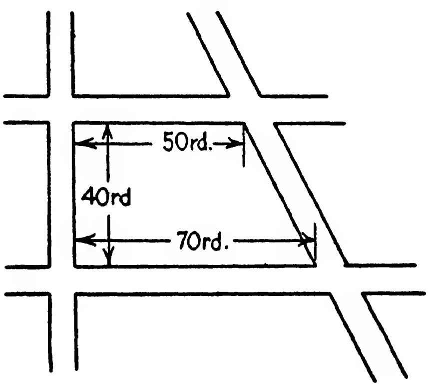


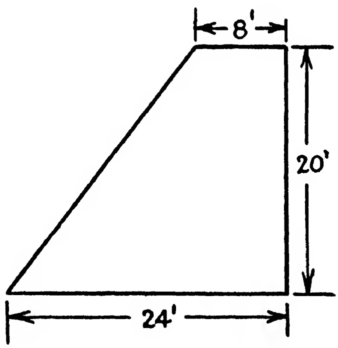
9. Find the total lateral surface of this 8-sided church steeple.
10. A steel plate riveted to a girder has the shape and dimensions shown. Find its area.



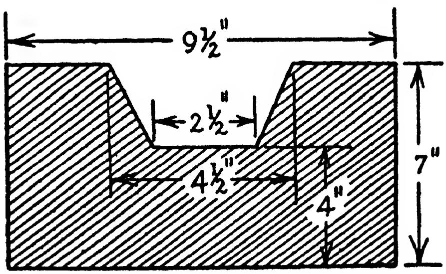




11. A plot of land is bounded by intersecting roads as shown. (a) How many square rods does the plot contain? (b) How many acres is this? (One acre=160 sq. rd.)
12. The cross section of the dam of a reservoir has the dimensions shown. What is the surface area of the end of the dam?

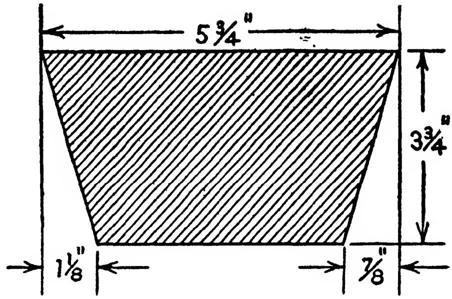


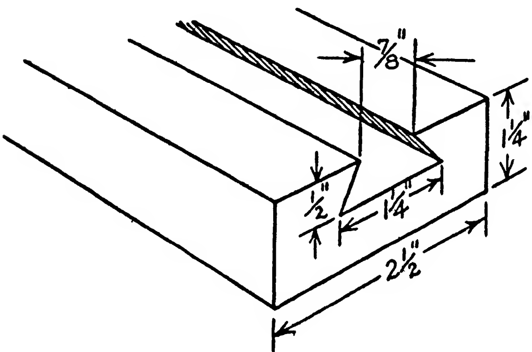




13. Find the area of the cross section of the steel channel beam shown.



14. The cross section of a fixture on a lathe is a trapezoid with the dimensions shown; what is the area of its face?



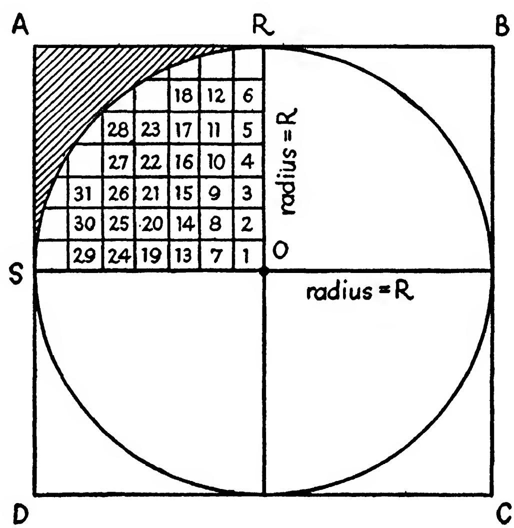




15. Find the cross-sectional area of the wooden dovetail shown in the figure.

**Area of a Circle.** The area included by a circle is found by squaring the radius and multiplying by $3\frac{1}{7}$; i.e.,

$$A=\pi R^2, \text{ or, since } R=\frac{D}{2},\ \ A=\frac{\pi D^2}{4}=.7854D^2.$$

Each side of the square is 14 units long; so is the diameter. The radius is 7 units in length. Now count the number of small squares in the quadrant ORS. There are 31 whole squares. Try to estimate the equivalent number of whole squares in the remaining imperfect squares. If you are careful, you will probably estimate 7 or 8 more; if you are very skillful, you will obtain approximately 7½ more, or about 38½ in all. Now, since a



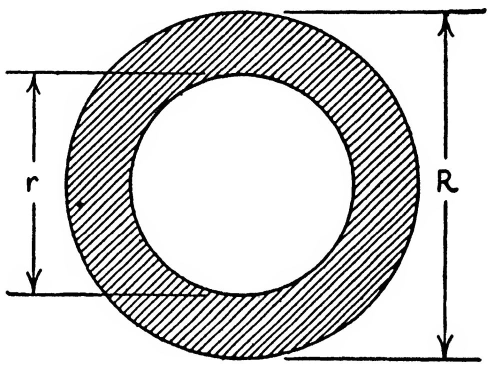
quadrant is ¼ of a circle, the entire circle contains $4 \times 38\frac{1}{2}$, or about 154 square units. Let's call the area of the circle A. Since R=7, then $R^2$=49. By counting the small squares, we found that A=154. Now divide 154 by 49:

$$\frac{A}{R^2} = {}^{154}\!/_{49} = 3\tfrac{1}{7} = \pi, \text{ showing}$$

$$\text{that } \frac{A}{R^2} = \pi, \text{ or } A = \pi R^2, \text{ as above.}$$

**Area of a Ring.** The area included between two concentric circles is called a *ring,* and is obviously found by deducting the area of the smaller circle from that of the larger, i.e.,

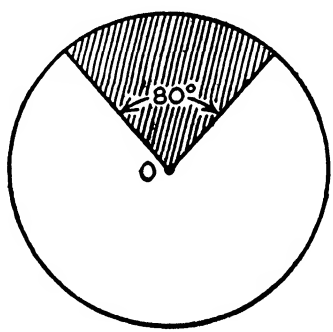
$$A = \pi R^2 - \pi r^2, \text{ or}$$
$$A = \pi(R^2 - r^2).$$




**Area of a Sector.** A sector of a circle is a figure bounded by a central angle and its subtended arc. The area of a sector (like the length of an arc) may easily be found by proportion; for

$$\frac{\text{Area of Sector}}{\text{Area of Circle}} = \frac{\text{Central Angle}}{360°};$$

$$\text{or, Sector} = \frac{\text{Central Angle}}{360°} \pi R^2.$$

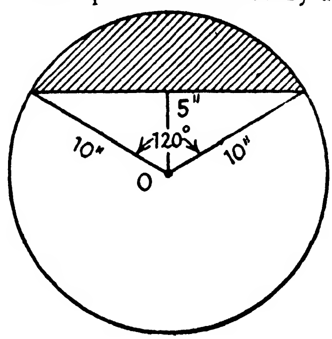



EXAMPLE: Find the area of an 80°-sector in a circle with a diameter of 7 inches.

SOLUTION: $R = 3\frac{1}{2}$

Sector $= (^{80}\!/_{360})\ (^{22}\!/_{7})\ (^{7}\!/_{2})\ (^{7}\!/_{2}) =$
8.56 sq. in., *Ans.*

**Area of a Segment.** A segment of a circle is that portion bounded by a chord and its subtended arc. Hence its area may be found by deducting the area of the triangular portion from the entire area of the corresponding sector.

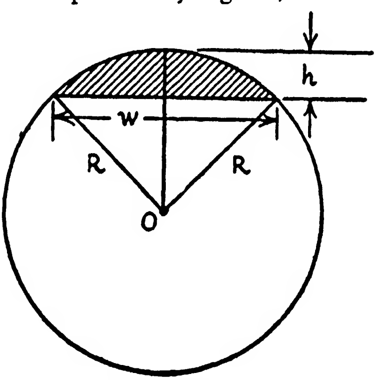



EXAMPLE: In a circle with a 10″ radius, find the area of a segment whose arc is 120°.

SOLUTION:

$$\text{Sector} = (^{120}/_{360})(3.14)(100) = 104.7$$
$$\text{Triangle} = {}^{100}/_{4}\sqrt{3} = 43.3$$
$$\text{Segment} = \text{Sector} - \text{Triangle} = 61.4 \text{ sq. in., } \textit{Ans.}$$

If the span $w$ only is given, as well as the radius, but not the central angle, the procedure is as follows: First the value of the height $h$ is determined from either of the following formulas, whichever is more convenient:



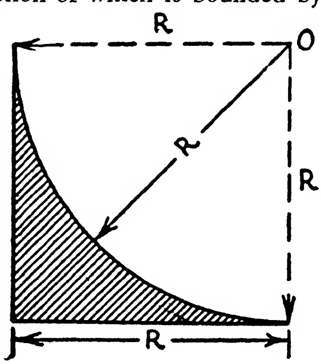

(1) $h = R - \sqrt{R^2 - (\frac{1}{2}w)^2}$

(2) $h = \frac{1}{2}(D - \sqrt{D^2 - w^2})$

The area may then be found by using the formula:

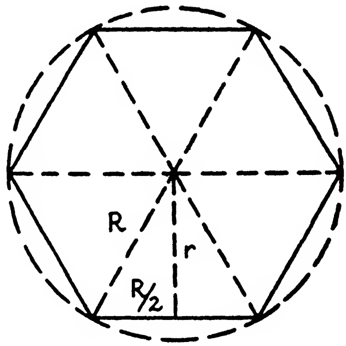
$$A = \frac{h(\frac{4}{3}w^2 + h^2)}{2w}$$

A *fillet* is a structural piece, the cross section of which is bounded by two adjacent sides of a square and the quadrant whose center is the opposite vertex of the square, as shown in the accompanying figure. The area of such a cross section is simply obtained by subtracting the area of the sector, which is one-fourth that of the circle, from the area of the square; thus



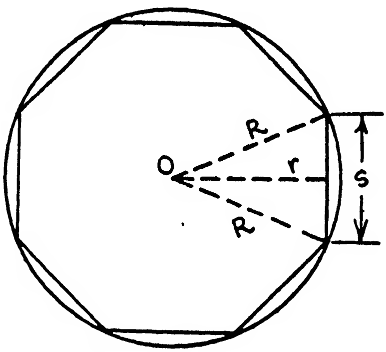

$$A = R^2 - \frac{1}{4}\pi R^2.$$

**Area of Regular Polygons.** The area of the square and the equilateral triangle have already been discussed. The only other regular polygon frequently encountered in the machine shop is the regular hexagon. Its area is simply six times that of one of the six congruent equilateral triangles of which it is made up, each side of the regular hexagon being equal to the radius of the circumscribed circle; thus




$$A=\frac{3R^2\sqrt{3}}{2}.$$

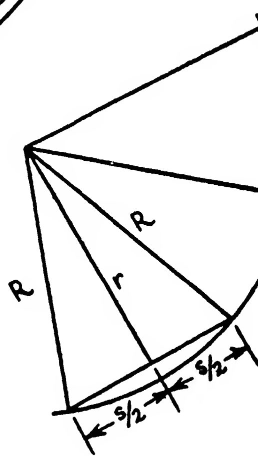
A general way of expressing the area of any regular polygon is in terms of its sides and apothem: $A=\frac{1}{2}(ns)r=Pr$,




where $n$=number of sides, $s$=length of each side, and $r$=the apothem. Of course, if any two of the measurements $R$ (radius), $r$ (apothem) and $s$ (length of side) are known, the third could always be found by the right-triangle rule. This is not always too convenient, however, and trigonometric methods are more practical, as will be seen later. Thus, since

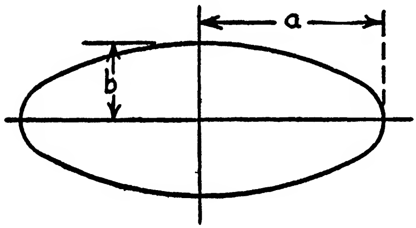
$$r=\sqrt{R^2-\frac{s^2}{4}}, \text{ then}$$

$$A=\frac{1}{2}(ns)\sqrt{R^2-\frac{s^2}{4}}.$$

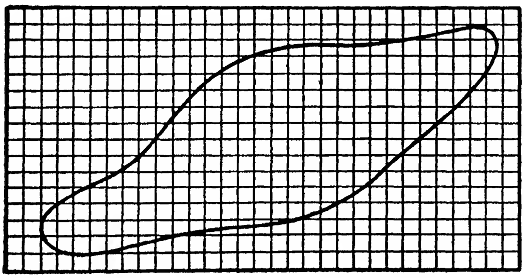


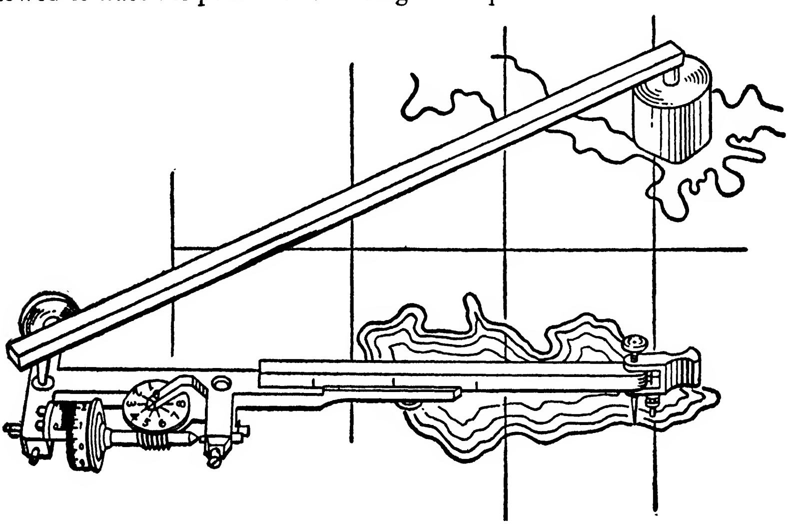

$$R^2=r^2+\left(\frac{s}{2}\right)^2$$

**Area of an Ellipse.** The area enclosed by an ellipse is found by multiplying the product of the major axis by the minor axis by $\pi$; or, $A=\pi ab$. If $a=b$, the ellipse becomes a circle, and $A=\pi aa=\pi a^2$, as already seen.

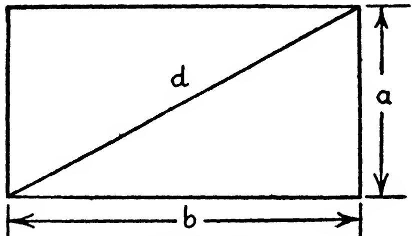



**Irregular Areas.** Not infrequently it is desired to obtain the area of an irregularly shaped figure, such as a plot of ground, the cross section of a specimen of material, a steam engine indicator, etc. Two methods are available in such cases: (1) an approximate value of the area may be found by drawing the figure to scale against a ruled area, and then counting and estimating the number of square units included within the area. If a more accurate value of the area is required, a *planimeter* may be used. This is a mechanical device which computes the area automatically as the needle of the instrument is allowed to trace the perimeter of the figure in question.



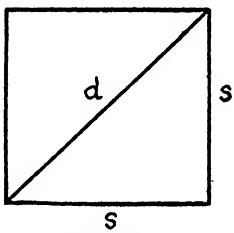


## SUMMARY OF FORMULAS



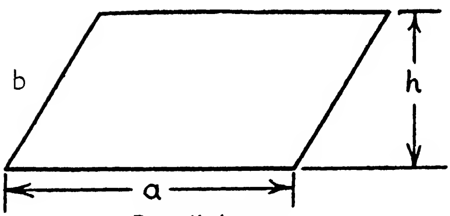
Rectangle

$P=2(a+b)$
$A=ab$
$d=\sqrt{a^2+b^2}$



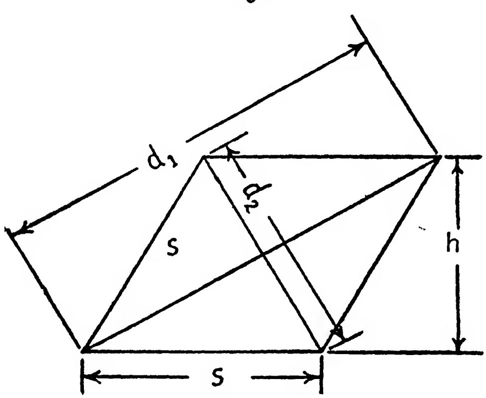
Square

$P=4s$
$A=s^2$
$d=s\sqrt{2}$
$s=\frac{1}{2}d\sqrt{2}$



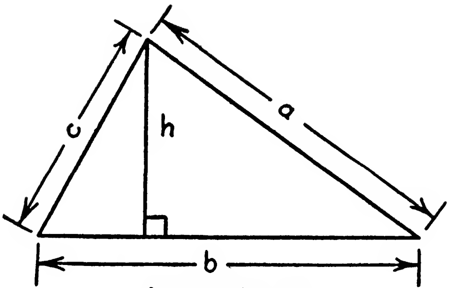
Parallelogram

$P=2(a+b)$
$A=ah$



Rhombus

$P=4s$
$A=sh$
$A=\frac{1}{2}d_1d_2$



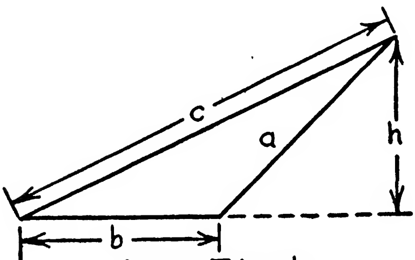

Acute Triangle

$P = a + b + c$

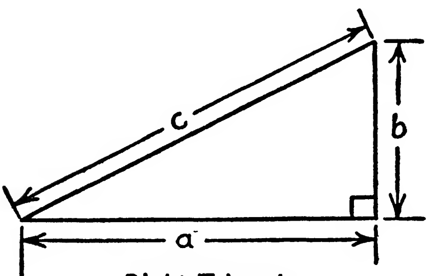
$A = \frac{1}{2}bh$

$A = \sqrt{s(s-a)(s-b)(s-c)}$

where $s$ = semiperimeter




Obtuse Triangle




Right Triangle

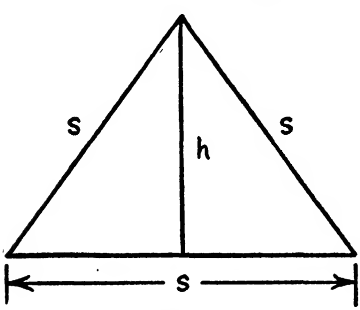
$P = a + b + c$

$A = \frac{1}{2}ab$

$c = \sqrt{a^2 + b^2}$

$a = \sqrt{c^2 - b^2}$

$b = \sqrt{c^2 - a^2}$

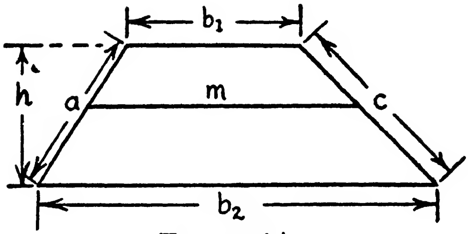



Equilateral Triangle

$P = 3s$

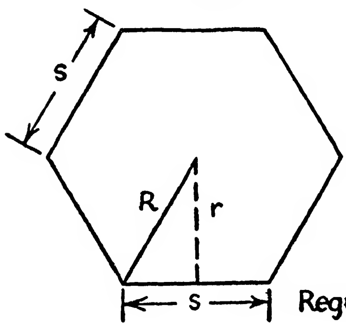
$A = \frac{s^2}{4}\sqrt{3}$

$h = \frac{s}{2}\sqrt{3}$



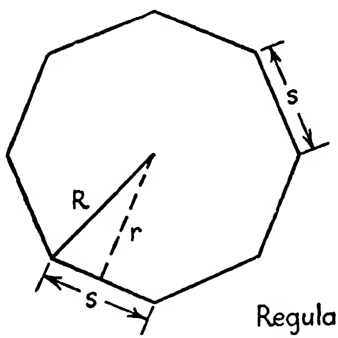

Trapezoid

$P=a+c+b_1+b_2$
$A=\frac{1}{2}h(b_1+b_2)$
$m=\frac{1}{2}(b_1+b_2)$
$A=\frac{1}{2}mh$



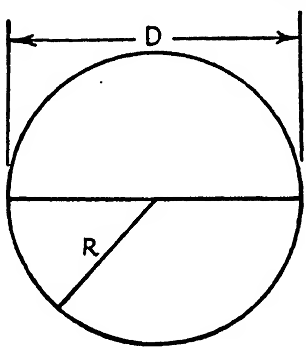

Regular Hexagon

$P=6s$
$R=s$
$r=\frac{s}{2}\sqrt{3}$
$A=\frac{3s^2}{2}\sqrt{3}$



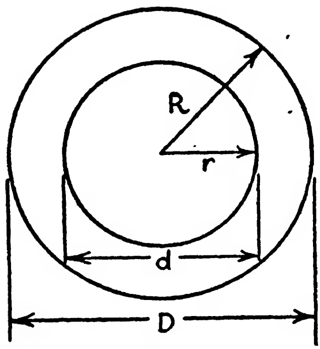

Regular n-gon

$P=ns$
$A=\frac{1}{2}Pr=\frac{1}{2}nsr$
$A=\frac{1}{2}(ns)\sqrt{R^2-\frac{s^2}{4}}$
$r=\sqrt{R^2-\frac{s^2}{4}}$



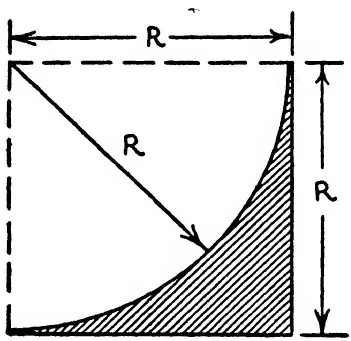

Circle

$C=2\pi R=\pi D$
$A=\pi R^2=\frac{\pi D^2}{4}=.7854D^2$



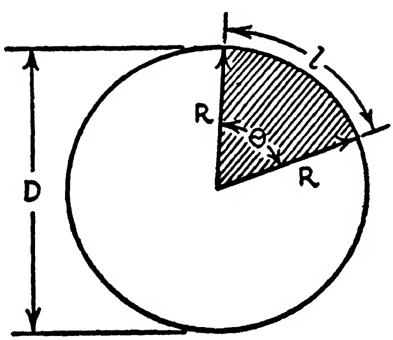

Ring

$$A=\pi(R^2-r^2)$$
$$A=\pi(R+r)\ (R-r)$$
$$A=\frac{\pi}{4}(D+d)\ (D-d)$$



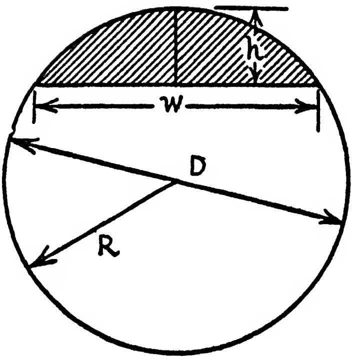

Fillet

$$P=2R+\frac{\pi R}{2}$$
$$A=R^2-\frac{\pi R^2}{4}$$



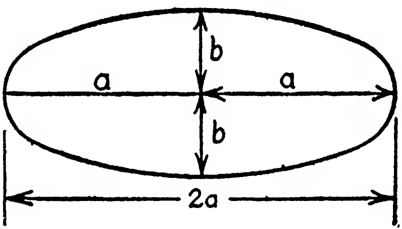

Sector

$$l\ (\text{arc})=\frac{\pi R\theta}{180}=\frac{\pi D\theta}{360}$$
$$A=\frac{\theta}{360}(\pi R^2)$$
$$A=\frac{\theta}{360}\left(\frac{\pi D^2}{4}\right)$$




Segment

$$h = R - \sqrt{R^2 - (\tfrac{1}{2}w)^2}$$
$$h = \tfrac{1}{2}(D - \sqrt{D^2 - w^2})$$
$$A = \frac{h(\tfrac{4}{3}w^2 + h^2)}{2w}$$

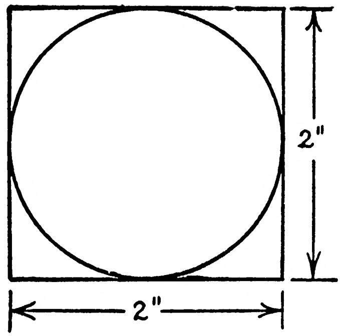



Ellipse

$P = \pi\sqrt{2(a^2 + b^2)}$ (approximately)
$A = \pi ab$

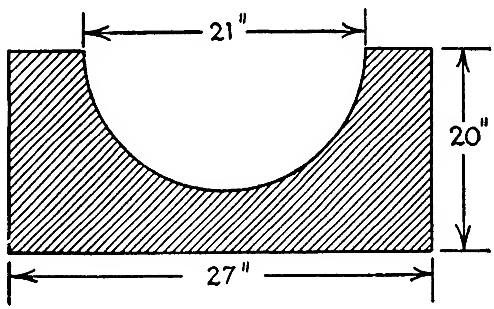
*Exercise 65.*

1. A circular-shaped rod is to be milled from a piece of square stock 2″ on a side. Find the area of the cross section of the finished rod.




2. Find the side of a square equivalent to a circle one foot in diameter.

3. Find the cross-sectional area of the semicircular wooden trough shown in the diagram.



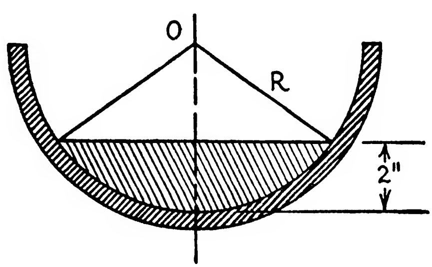

4. If the diameter of a circle is 40″, what is the area of a segment whose arc is 120°?

5. The diameters of two circles are 4″ and 6″, respectively. Find (a) the ratio of their circumferences; (b) the ratio of their areas.

6. What per cent of any circle is wasted if the largest possible hexagon is cut out of it?

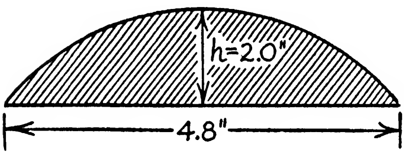
7. Find the cross-sectional area of the metal in a hollow shaft measuring 3½″ outside diameter and 2″ inside diameter.

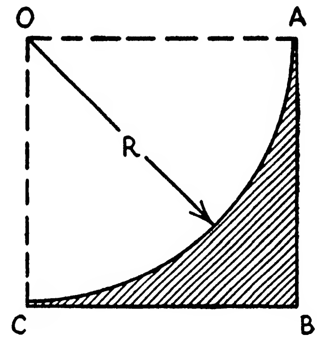
8. The inside diameter of a horizontal pipe is 10½″. Water is standing to a depth of 2″. Find the cross-sectional area of water in the pipe.



9. A wooden cylinder with a circular cross section is to be planed down to have the largest possible square cross section. If the diameter of the original rod is 3½″, what per cent is wasted in shavings?

10. A metal stamping has the shape of a circular segment with a span of 4.8 in. and a height of 2.0″. Find its area.



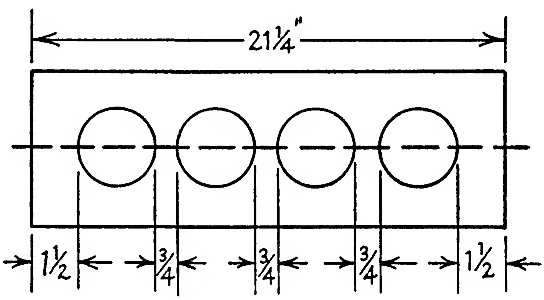




11. Find the area of the shaded fillet shown, where the radius of the fillet is 4″, and OABC is a square.

12. Find the area of a segment having for its chord a side of a regular inscribed hexagon, if the radius of the circle is 10½″.

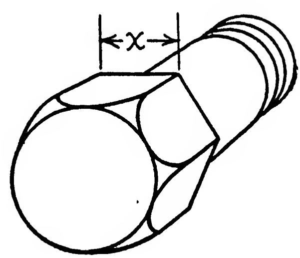
13. The end section of a gasoline tank is an ellipse with a major axis of 28″ and a minor axis of 22″. Find the area of this cross section of the tank.

14. The gasket of an auto engine is as shown; find the area of the metal wasted when the holes are punched out.



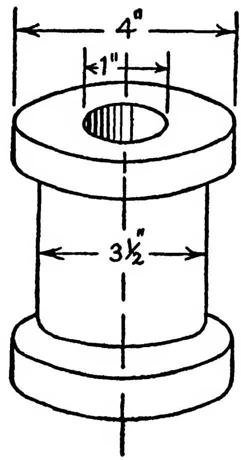

15. In a steam engine having a piston 18″ in diameter, the pressure upon the piston is 84 lb. per sq. in. Find the total pressure on the piston.

16. If $x = 1½″$, what is the diameter of the round stock needed to make the hexagonal head of this machine stud?

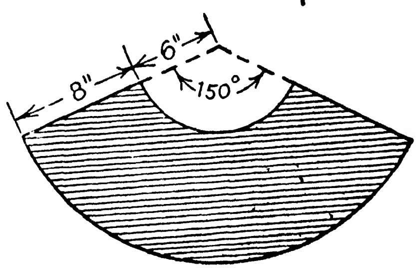


17. If a $3\frac{1}{2}''$ circular piece of rubber is cut from a sheet that is $4''$ square, what per cent of the material is left?

18. An $8''$-pipe supplies only $\frac{2}{3}$ the amount of water required in a given time. Assuming that the flow is proportional to the cross-sectional area, find the diameter of the smallest pipe that will supply the required amount, if the pipe sizes come only in diameters of whole numbers of inches.



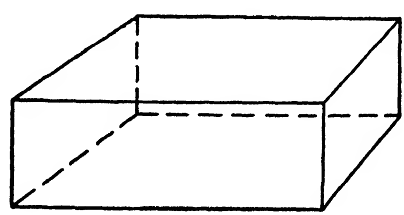

19. Find the area of the end of this spool before the hole is drilled; after the hole is drilled.



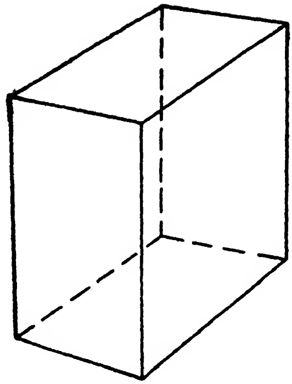

20. A parchment lampshade is to be made from a piece of material cut according to the pattern shown; find the area of the parchment required.

## 18. MEASUREMENT OF SOLID FIGURES

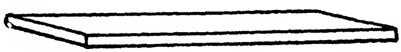
**Rectangular Solids.** A *rectangular solid* is a solid figure with 6 *faces* and 12 *edges;* all of the faces are rectangles, being, in fact, three pairs of congruent rectangles, and all the edges are perpendicular to the faces which they meet. The six faces are usually referred to as the "top and bottom," or the *bases;* as the "front and back,"

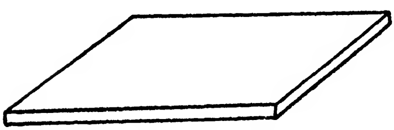


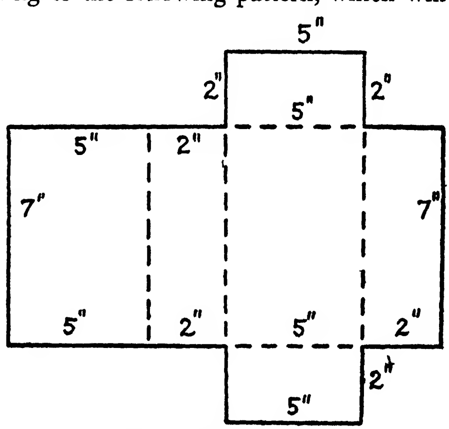
and as the "two ends," the last four constituting the *lateral* faces. Such figures are also known as rectangular *prisms,* but there are other types of prisms, too, as will be seen shortly. The three different edges are often known as the *length, width* and *height* (or thickness), although it is sometimes better to refer to the perpendicular distance between the bases as the *altitude* instead of the height or thickness.



Rectangular-solid shapes are met in shop practice in various proportions. Thus metal bars, planks, and joists are usually rectangular solids with one of the three "dimensions" much greater than the other two, as suggested in the diagram; thus we speak of a "two by four" which is 8 ft. long, meaning a joist 2 in.$\times$4 in.$\times$8 ft. Or again, in machine-shop work, frequent use is made of flat, rectangular *plates* and *sheets* of metal; if rectangular "pieces" are cut out, they are still rectangular solids, no matter how thin. A rectangular solid may be "developed" according to the following pattern, which will help to understand its geometric properties. In other words, by cutting along the solid lines, and folding along the dotted lines, a rectangular solid would be formed.








**Area and Volume of a Rectangular Solid.** The *lateral area* of a rectangular solid with dimensions $l$, $w$, and $h$ is equal to $2lh+2wh$,

or, $L.A.=2h(l+w)$,

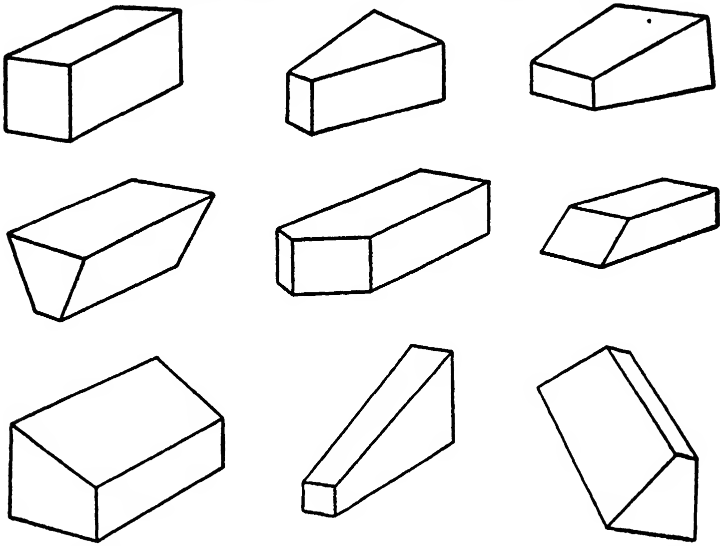
or, $L.A.=ph$, where $p=$ the perimeter of the base.

The *total area* is the lateral area plus the area of the two bases, or

$$T.A.=2lh+2wh+2wl.$$

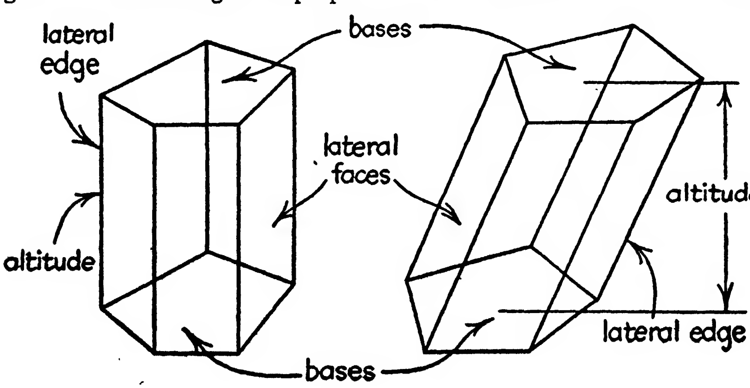
The *volume* of a rectangular solid, which has already been discussed, equals the product of its three dimensions; thus $V=lwh$,

or $V=Ah$, where $A$ is the area of either base.

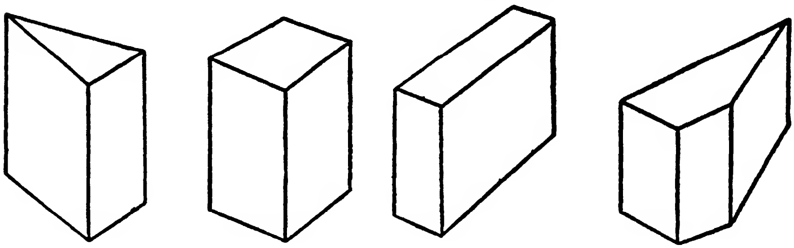


TYPICAL SHAPES OF FIRECLAY AND SILICA BRICK
(for high temperature work in the construction of ovens, furnaces and kilns.)

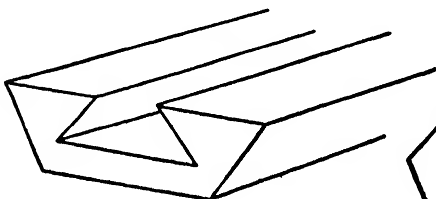
**Prisms.** Solids bounded partly by a series of flat surfaces (faces) whose intersections are parallel lines, and partly by a pair of additional parallel surfaces (bases) each of which intersects the series of lateral faces are called *prisms.* Such solids may be *right prisms* or *oblique prisms,* according as their *lateral edges* are perpendicular to the *bases* or not. The lateral



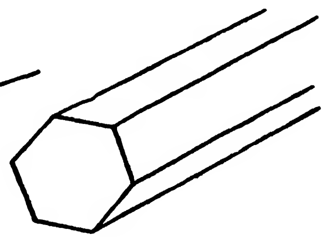
faces of a prism are always parallelograms; they may or may not be congruent, and they may or may not be rectangles. But their lines of intersection are always parallel, whether oblique or perpendicular to the bases. The bases are congruent polygons, parallel to one another, and each base intersects all the lateral faces. These bases may be triangles, squares,



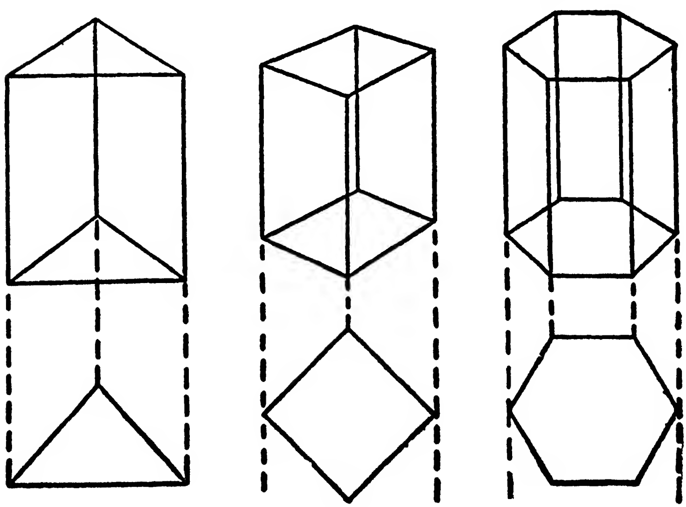
Triangular Prism Square Prism Rectangular Prism Trapezoidal Prism



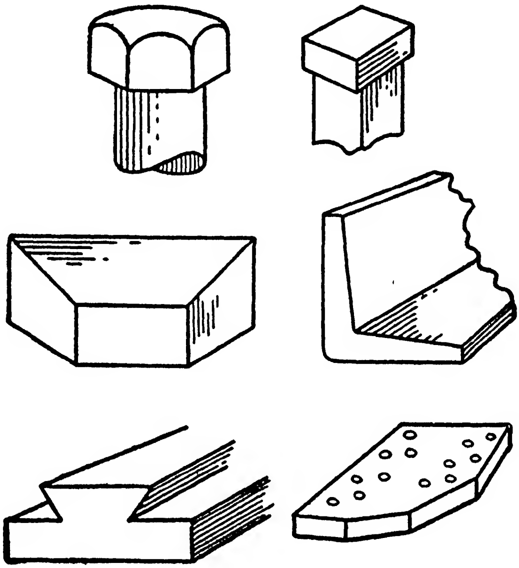
Grooved Prism



Hexagonal Prism

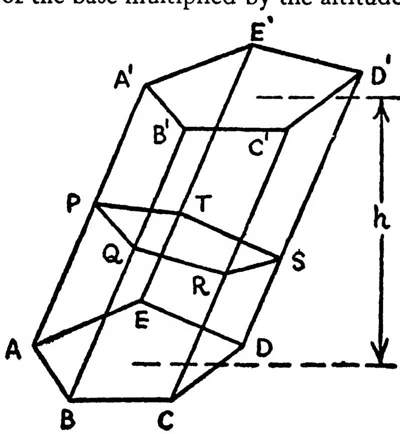


hexagons, etc. The *altitude* of any prism is the perpendicular distance between the two bases. Thus in a right prism the altitude is equal in length to any lateral edge, since all the lateral edges are perpendicular to the bases; in an oblique prism the altitude is less than the length of a lateral edge. In both types of prisms, however, all the lateral edges are equal in length.

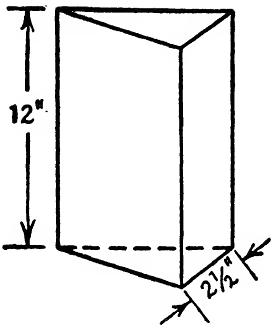


A *regular prism* is any right prism whose bases are regular polygons. In such prisms every non-lateral edge, i.e., base edge, equals every other non-lateral edge. In connection with mechanical work, many parts, such as strips, plates, beams, bars, sheets, angle irons, and the like are in the shape of regular prisms, or partly so, at any rate.

The *lateral area* of any regular prism equals the perimeter of the base times the altitude; the *total area* then equals the lateral area plus twice the area of the base. The *volume* of any regular prism equals the area of the base multiplied by the altitude. For specific types of prisms see the diagram above, or the summary of formulas near the end of this section. If, on the other hand, the figure is an oblique prism, its lateral area can be found by computing the area of each face separately, or by multiplying the perimeter of a right section by the length of a lateral edge; or $L.A. = (PQ+QR+RST+\cdots)\times AA'$. The volume is still equal to the area of $(ABCDE)\times h$.




EXAMPLE 1: Find the volume of a right prism whose base is an equilateral triangle $2\frac{1}{2}''$ on a side and whose altitude is $12''$.




SOLUTION:

$$V=Bh;\ B=\frac{s^2}{4}\sqrt{3}$$

$$B=\left(\frac{1}{4}\right)\left(\frac{25}{4}\right)\sqrt{3}=\frac{25}{16}(1.73)=2.70$$

$$V=Bh=(2.7)(12)=32.4 \text{ cu. in., } \textit{Ans.}$$

EXAMPLE 2: What is the lateral area of a regular hexagonal prism, ¾" on a lateral edge, and 3" on each edge of the base?

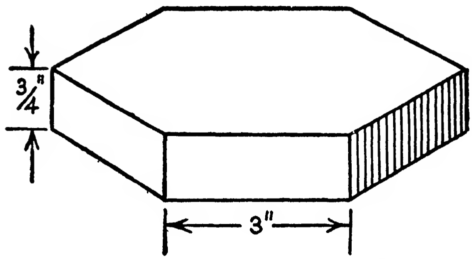
SOLUTION:

$$L.A.=(6)(3)\left(\frac{3}{4}\right)=13.50$$

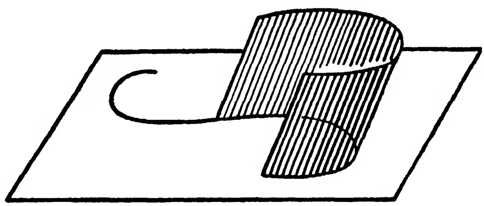
Area of base=

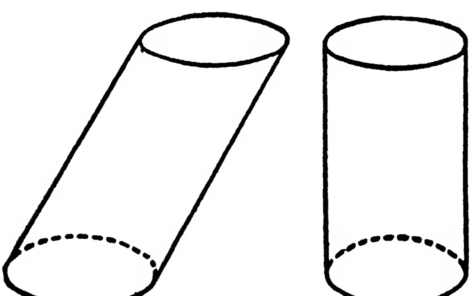
$$\left(6\right)\left(\frac{3^2}{4}\right)\sqrt{3}=23.36$$

$$T.A.=13.50+(2)(23.36)$$
$$=60.2 \text{ sq. in., } \textit{Ans.}$$



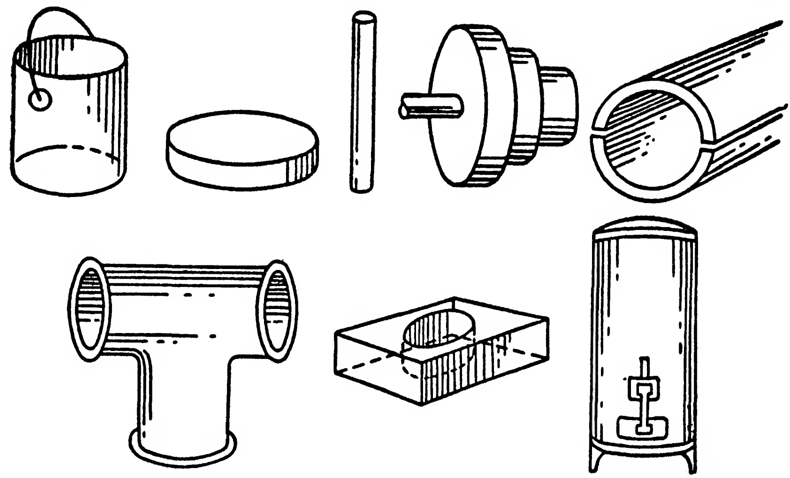

**Cylinders.** A *cylindrical surface* is one that has been generated by a straight line moving in such a way that it follows a fixed curve in a plane, while maintaining a constant angle to the plane. If the fixed curve (directrix) whose path is followed is a circle, the surface generated is a circularly cylindrical surface; if in addition the "generating line" remains perpendicular to the plane, it is a *right* circular cylindrical surface. A solid formed by such a surface *and* two parallel planes intersecting the cylindrical surface is called an *oblique circular cylinder* or a *right circular cylinder,* respectively. An approximate illustration of an oblique elliptical cylinder would be the shape of some steamship funnels.





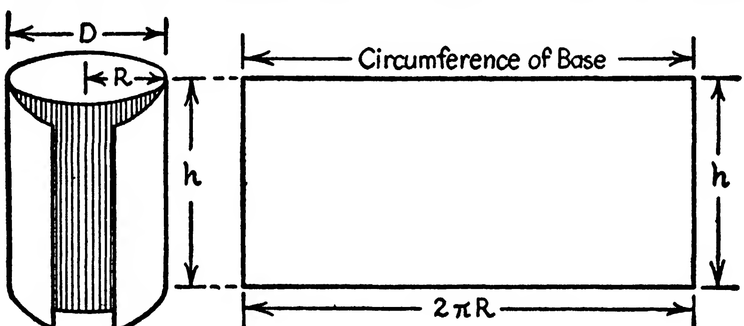
**Right Circular Cylinders.** Many common objects are

in the form of a right circular cylinder: spools, drums, tubes, pipes, discs, pulleys, wheels, containers, cans, etc. In a right circular cylinder, the bases are equal circles, there are no "lateral edges," and the altitude is the perpendicular distance between the two bases. If the altitude of a right



circular cylinder is relatively small compared to its diameter, it is usually called a *disc;* if the altitude is considerably longer than its diameter, it is called a *rod* or a *tube,* according to whether it is solid or hollow.

The lateral area of a right circular cylinder will be seen from the fol-




lowing diagram to be given by

$$L.A.=2\pi Rh=\pi Dh;$$

and the total area is

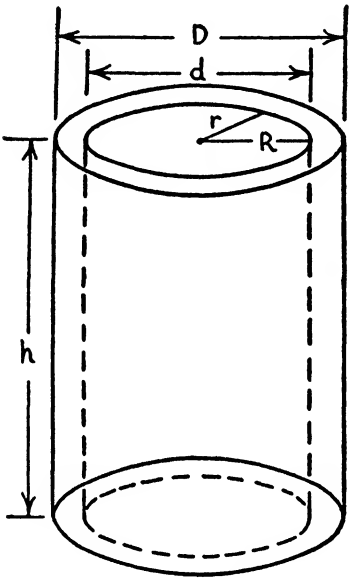
$$T.A.=2\pi Rh+2\pi R^2$$

$$\text{or, } T.A.=2\pi R(h+R)$$

$$=\pi D\left(h+\frac{D}{2}\right)$$

The volume of a right circular cylinder, like that of a right prism, equals the area of the base multiplied by the altitude; or

$$V=\pi R^2h=\frac{\pi D^2}{4}h.$$




**Hollow Cylinders.** For such figures the following formulas will be found useful:

$$L.A.\ (inside)=2\pi rh$$
$$L.A.\ (outside)=2\pi Rh$$
$$End\ area\ (one)=\pi(R^2-r^2)$$
$$V=\pi h(R^2-r^2)$$
$$=\frac{\pi h}{4}(D^2-d^2)$$

**Fillet.** The volume of a fillet is given by the product of its cross-sectional area multiplied by its length (or height).

EXAMPLE 1: Find the contents in gallons of a cylindrical tank 7 ft. in diameter and 14 ft. high, if filled to a level 9½ ft. above the bottom.

SOLUTION: $V=\pi R^2h$

$$=\left(\frac{22}{7}\right)\left(\frac{49}{4}\right)\left(\frac{19}{2}\right)=365.75 \text{ cu. ft.}$$

1 cu. ft.=7½ gal.

365.75×7½=2743 gal., *Ans.*

EXAMPLE 2: What is the area of the inside surface of an open cylindrical tank measuring 3½ ft. in height and 1½ ft. in diameter?

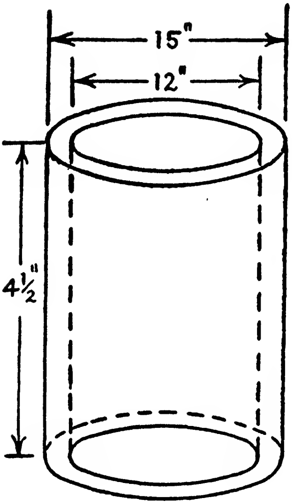
SOLUTION: $L.A.=2\pi Rh$

$$=2\left(\frac{22}{7}\right)\left(\frac{3}{4}\right)\left(\frac{7}{2}\right)=16.5 \text{ sq. ft.}$$

$B=\pi R^2h$

$$=\left(\frac{22}{7}\right)\left(\frac{9}{16}\right)\left(\frac{7}{2}\right)=6.2 \text{ sq. ft.}$$

Total area=22.7 sq. ft., *Ans.*




EXAMPLE 3: Find the weight of a section of cylindrical metal casting 4½ ft. long, with an inside diameter of 12″ and an outside diameter of 15″, if the material weighs 0.24 lb. per cu. in.

$$V=\frac{\pi h}{4}(D^2-d^2)$$

$$V=\left(\frac{22}{7}\right)\left(\frac{9}{2}\right)\left(\frac{12}{4}\right)(225-144)$$

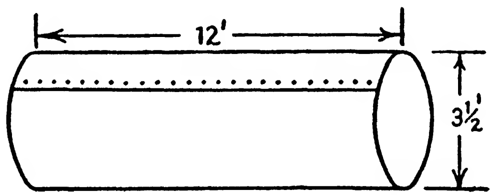
$V$=3435 cu. in.

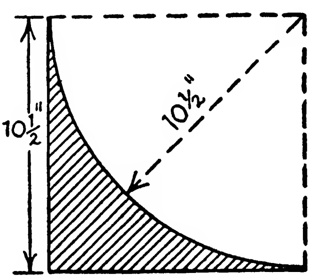
Weight=3435×0.24=824 lb., *Ans.*

*Exercise 66.*

(*For reference:* 1 gal.=231 cu. in.; 1 cu. ft.=7½ gal.; 1 cu. ft. water=62½ lb.)

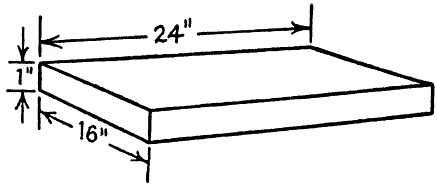
1. What is the weight of a rectangular steel bar 48″×4″×1½″, if the metal weighs .25 lb. per cu. in.?
2. Allowing 4″ for overlapping at the seam along its entire length, how many sq. ft. of galvanized iron are required for this section of metal pipe?



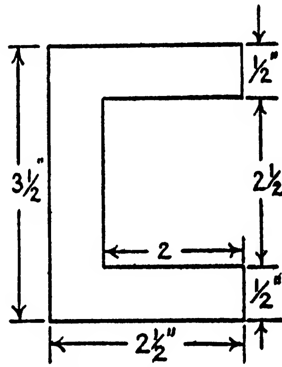




3. A fillet with the cross section shown is 14 ft. in length. How many cu. ft. of material are there in 20 such pieces?
4. Brass discs ½″ thick are to be machined from 4″ diameter stock. (a) How many cu. in. of metal are there in each disc? (b) What is the weight of each disc if brass weighs .31 lb. per cu. in.?
5. An irregular piece of metal is placed in a cylindrical vessel 3 in. in diameter and partly full of water. If the water rises 3½ in., what is the volume of the piece of metal?

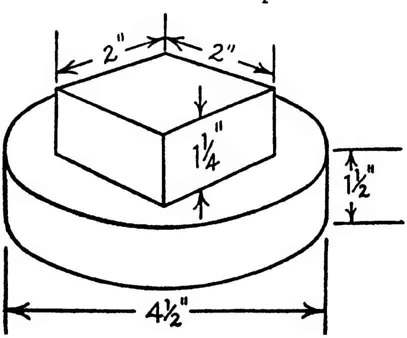
6. At 40¢ per lb., what is the cost of copper required to make 24 open pans with the dimensions shown, if sheet copper weighs .8 lb. to the square foot, and 10% is allowed for lapping and waste?

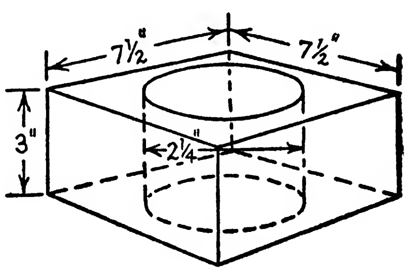



7. Find the weight of a cast-iron cylinder 2.80 in. in diameter and 2 ft. long, if cast iron weighs 448 lb. per cu. ft.
8. Hollow metal bars in the shape of a regular hexagon in 12 ft. lengths are to be plated with a special chromium finish. If the cross section of the bar is a regular hexagon ¾″ on a side, find the total surface to be plated on 100 such bars.
9. Laminations for an electrical machine are made of pieces of sheet iron with the shape and dimensions shown. What is the weight of 1000 such pieces, if the iron used weighs .48 lb. per sq. ft.?



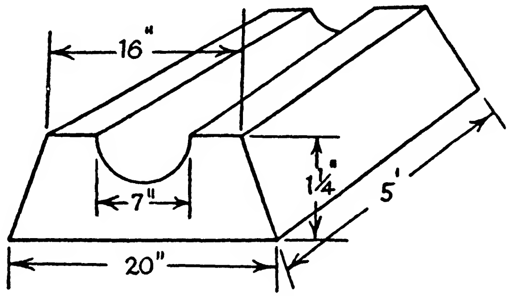

10. A cylindrical hot water boiler has a diameter of 14″ and stands 4′ 8″ high. How many gallons will it hold when full?
11. A rectangular block of gold metal measures 10″×8″×2″. If it is hammered and rolled into sheets 8″ square and .02″ thick, how many such sheets will be obtained?
12. Find the volume of a cylindrical bronze ring having an inside diameter of 4⅜″ and an outside diameter of 6¼″, if it is ¾″ thick.
13. Find the volume of the solid iron plug with dimensions as shown.
14. A circular hole is drilled through the metal block as shown; what is the volume of the piece after the hole has been drilled?



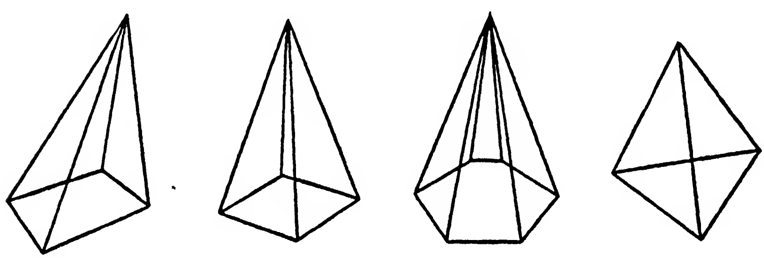
15. A rod of copper 14 ft. long and 2 sq. in. in cross section is melted and formed into a wire ⅛ in. in diameter. Find the length of the wire.

16. Concrete blocks like the one shown in the sketch are made in molds. If the sections are 5 ft. long and 7¼ in. high, how many cu. yd. of concrete are required for 48 such sections?



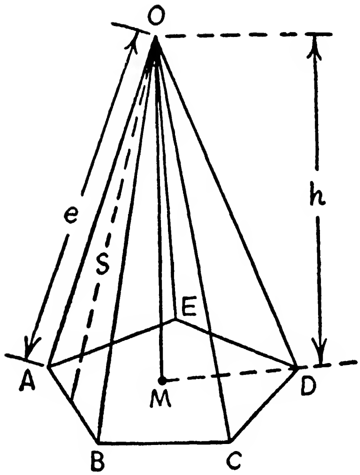

17. If 2 gallons of water are poured into a cylindrical jar 7 inches in diameter, how high will the level of the water rise?
18. Sections of reinforced concrete water main are hollow cylinders 12 ft. long and 4″ thick. If the inside diameter is 48″, and the concrete weighs 105 lb. per cubic foot, what is the weight of one such section?
19. A cylindrical paint can 14 in. high holds exactly 2 gallons. What is the diameter of the can?
20. In a building there are 2100 ft. of steam piping with a 12″ outside diameter. How many sq. ft. of pipe surface does this represent?
21. If a bar of metal 2″ in diameter weighs 10.2 lb. per foot of length, what is the weight per running foot of a bar 3″ square of the same material?
22. A tunnel whose cross section is a semicircle 21 ft. high is ¼ mile long. How many cu. yd. of earth were removed when it was excavated?

**Pyramids.** Any solid figure, one of whose faces is a polygon and whose other faces are all triangles having a common vertex, is a *pyramid.*



Oblique Pyramid — Regular Square Pyramid — Regular Hexagonal Pyramid — Regular Tetrahedron

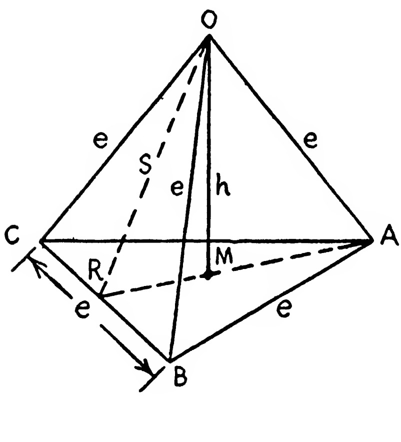
The *altitude* of any pyramid is the perpendicular distance from the




vertex (O) to the plane of the base. If the base is a regular polygon, and the altitude of the pyramid passes through the center of the base, the solid is a *regular pyramid.* In such a regular pyramid, all the lateral faces are obviously equal isosceles triangles; the altitude of any lateral face is known as the *slant height* of the pyramid, and is always greater than the altitude of the pyramid.

The lateral surface of a regular pyramid is given by $L.A.=\frac{1}{2}Ps$, where $P$=perimeter of the base and $s$=slant height. The total area, of course, is the lateral area plus the area of the base. The volume of any pyramid, whether regular or not, equals ⅓ the area of its base multiplied by its altitude.

**Regular Tetrahedron.** A special case of a regular pyramid is the case of a regular triangular pyramid in which not only the base, but each of the three lateral faces, are all equilateral triangles. Such a figure is called a *regular tetrahedron,* since all four faces are identical. If each of the six equal edges is denoted by $e$, then careful study of the figure will show that the following relations hold:




(1) $OR=s=\frac{e}{2}\sqrt{3}$

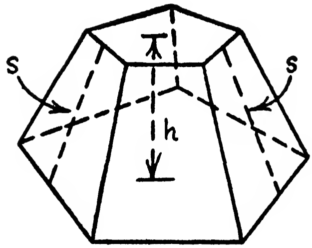
(2) $AR=OR=\frac{e}{2}\sqrt{3}$

(3) $AM=\frac{2}{3}(AR)=\frac{e}{3}\sqrt{3}$

(4) $OM=h=\sqrt{(OA)^2-(AM)^2}$

$$=\sqrt{e^2-\left(\frac{e\sqrt{3}}{3}\right)^2}=\frac{e}{3}\sqrt{6}$$

**Frustum of a Pyramid.** If a portion of a pyramid including the vertex is cut off by a plane parallel to the base, the part left is called the *frustum*

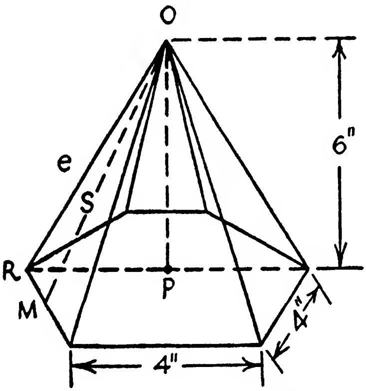



of the pyramid. Thus in the frustum of a regular pyramid each lateral face is an isosceles trapezoid; all these lateral faces are equal to one another, and the slant height of the frustum is the altitude of the trapezoid. The altitude of the frustum is the perpendicular distance between the two bases, which are similar regular polygons.

The lateral area of the frustum of a regular pyramid equals half the sum of the perimeters of the bases multiplied by the slant height, or

$$L.A. = \tfrac{1}{2}(P+p)s.$$

The volume of a regular frustrum is given by $V=\tfrac{1}{3}h(B+b+\sqrt{Bb})$, where $b$ and $B$ are the areas of the upper and lower bases, respectively.




Example 1: Find the lateral area and volume of a regular hexagonal pyramid whose altitude is 6″, if the sides of the base are 4″ each.

Solution:

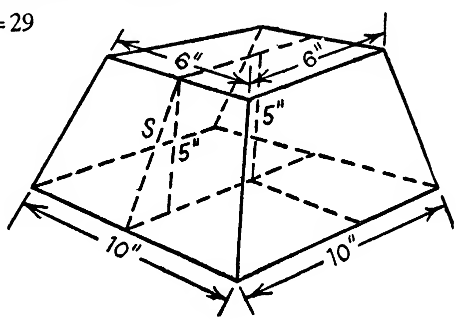
$$(OR)^2=(OP)^2+(PR)^2$$
$$e^2=4^2+6^2=16+36=52$$
$$(OM)^2=(RM)^2+(OR)^2$$
$$s^2=4+52=56$$
$$s=\sqrt{56}=7.5$$

$$L.A.=\tfrac{1}{2}ps=\tfrac{1}{2}(24)(7.5)=90.0+ \text{ sq. in., } Ans.$$
$$V=\tfrac{1}{3}Bh=\tfrac{1}{3}[(6)(4\sqrt{3})(6)]=83.0+ \text{cu. in., } Ans.$$

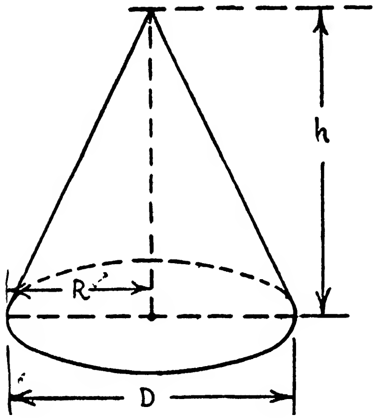
Example 2: Find the total area and volume of the frustum of a square pyramid with dimensions as shown.

Solution: $s^2=5^2+2^2=25+4=29$

$$s=\sqrt{29}=5.4$$
$$L.A.=\tfrac{1}{2}s(p_1+p_2)$$
$$=\tfrac{1}{2}(5.4)(24+40)$$
$$=172.8 \text{ sq. in.}$$
$$T.A.=172.8+36+100$$
$$=308.8 \text{ sq. in., } Ans.$$
$$V=\tfrac{1}{3}h(B_1+B_2+\sqrt{B_1B_2})$$
$$=\tfrac{1}{3}(5)(36+100+60)$$
$$=326.7 \text{ cu. in., } Ans.$$



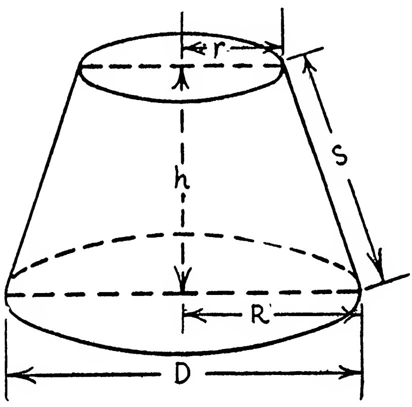
**Right Circular Cones.** The properties of right circular cones are analogous to those of regular pyramids, of which, indeed, the cone can be considered as the limiting case as the number of sides is increased indefinitely. Hence we have:




$L.A. = \frac{1}{2}\, Cs = \frac{1}{2}(2\pi R)s = \pi Rs$

$T.A. = \pi R^2 + \pi Rs = \pi R(R+s)$

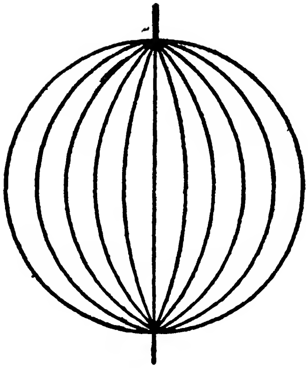
$$V = \frac{1}{3}Bh = \frac{1}{3}\pi R^2 h = \frac{\pi D^2 h}{12}$$

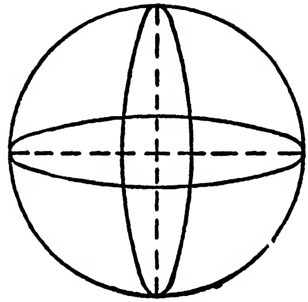


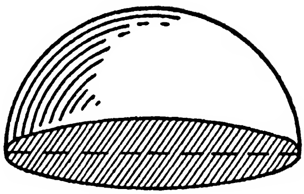

Similarly for the frustum of a regular cone:

$$L.A. = \frac{1}{2}(C+c)s$$
$$= \frac{1}{2}(2\pi R + 2\pi r)s$$
$$= \pi s(R+r).$$
$$T.A. = \pi s(R+r) + \pi R^2 + \pi r^2$$
$$= \pi[(R+r)s + R^2 + r^2]$$
$$V = \frac{1}{3}h[\pi R^2 + \pi r^2 + \sqrt{\pi R^2 + \pi r^2}]$$
$$= \frac{1}{3}\pi h[R^2 + r^2 + Rr]$$

**Sphere.** In practical use, the sphere or ball finds its commonest application where friction between moving parts is to be a minimum, as in ball bearings, or where the symmetry and "perfect beauty" of a sphere is attractive, as in ornamental parts. The sphere is a "closed" curved surface, every point on which is equally distant from a point within called the center. A sphere may also be thought of as the surface generated by revolving a semicircle about its diameter as an axis. A plane passed through the center of a sphere divides it into two equal *hemispheres*. The base of each hemisphere is a circle whose diameter is the same as that of the sphere; they are known as *great circles*.







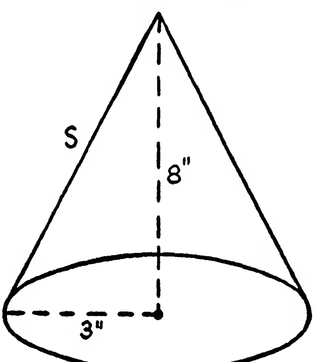
The surface of a sphere is found to be four times that of a great circle of the sphere, or

$$A = 4\pi R^2 = \pi D^2;$$

and the volume is given by

$$V = \tfrac{4}{3}\pi R^3 = \frac{\pi D^3}{6}.$$

Example 1: Find the total area and volume of a right circular cone if the altitude is 8″ and the diameter of the base is 6″.




Solution:

$$s = \sqrt{8^2 + 3^2} = \sqrt{73} = 8.6$$

$$L.A. = \pi Rs$$

$$= (\tfrac{22}{7})(3)(8.6) = 81.1 \text{ sq. in.}$$

$$T.A. = L.A. + (\tfrac{22}{7})(9)$$

$$= 81.1 + 28.3 = 109.4 \text{ sq. in., } \textit{Ans.}$$

$$V = \pi R^2 h = (\tfrac{22}{7})(9)(8) =$$

$$226.4 \text{ cu. in., } \textit{Ans.}$$

Example 2: Find the lateral area and volume of the frustum of a right circular cone whose altitude is 10″, and the diameters of whose bases are 8″ and 18″, respectively.

SOLUTION:

$s^2=10^2+5^2$

$s=\sqrt{100+25}=\sqrt{125}=11.2$

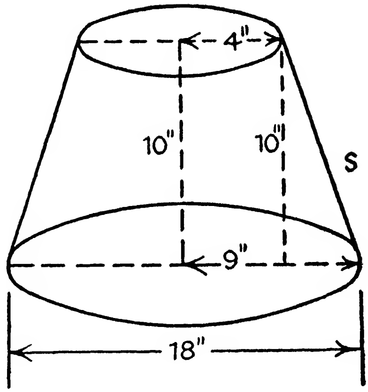
$L.A.=\pi s(R+r)$

$=(^{22}\!/_7)(11.2)(9+4)=$ 44.3 sq. in., *Ans.*

$V=\frac{1}{3}\pi h[R^2+r^2+Rr]$

$=(\frac{1}{3})(^{22}\!/_7)(10)[81+16+36]$

$=1393+$ cu. in., *Ans.*



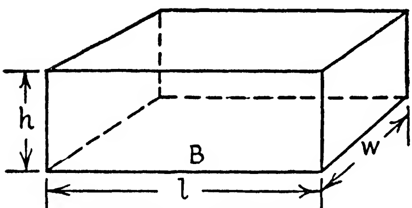

EXAMPLE 3: A hollow metal container is in the shape of a hemisphere whose diameter is 42 cm. What is its capacity?

SOLUTION: $V=\frac{4}{3}\pi R^3(\frac{1}{2})$

$=(\frac{4}{3})(^{22}\!/_7)(21^3)(\frac{1}{2})=19{,}404$ cu. cm., *Ans.*

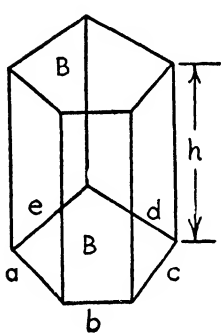
## SUMMARY OF FORMULAS

*Areas and Volumes of Solid Figures*



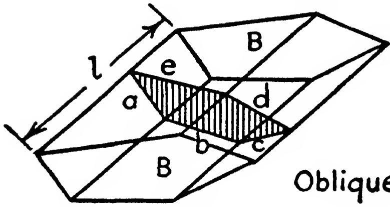

Rectangular Solid

$L.A.=ph=2h(l+w)$
$T.A.=2(lw+wh+wl)$
$V=lwh=Bh$



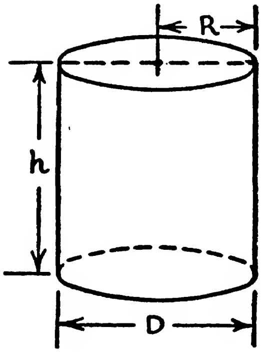

Right Prism

$L.A.=ph=h(a+b+c+\cdots)$
$T.A.=ph+2B$
$V=Bh$



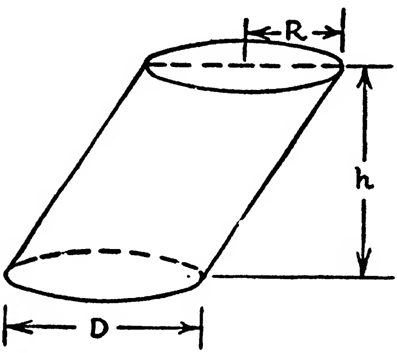

$$L.A. = pl = h(a+b+c+\cdots)$$
$$T.A. = pl + 2B$$
$$V = Bh$$

Oblique Prism



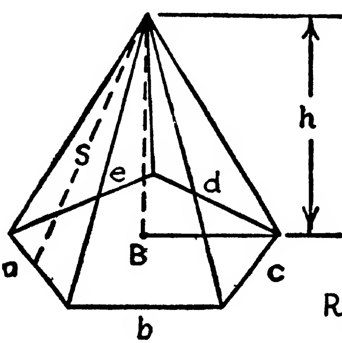

$$L.A. = 2\pi Rh$$
$$T.A. = 2\pi Rh + 2\pi R^2$$
$$= 2\pi R(h+R)$$
$$V = \pi R^2 h$$
$$= \frac{\pi h}{4} D^2$$

Right Circular Cylinder



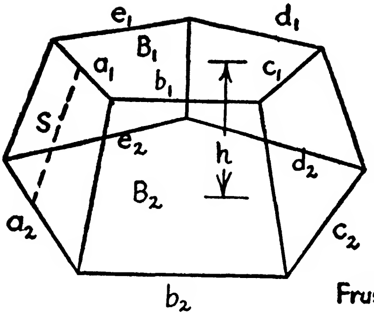

$$V = \pi R^2 h$$

Oblique Circular Cylinder



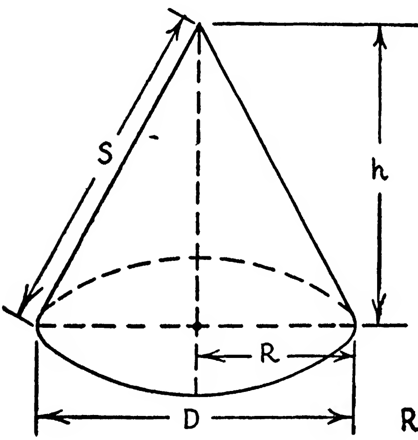

$$L.A. = \tfrac{1}{2}ps = \tfrac{1}{2}s(a+b+c+\cdots)$$
$$T.A. = \tfrac{1}{2}ps + B$$
$$V = \tfrac{1}{3}Bh$$

Regular Pyramid



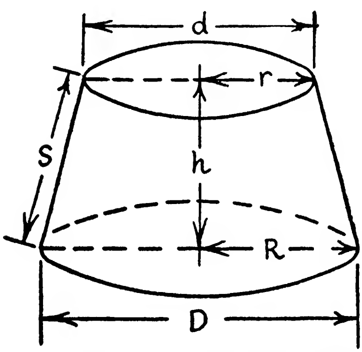

$$L.A. = \tfrac{1}{2}s(p_1 + p_2)$$
$$= \tfrac{1}{2}s[(a_1 + b_1 + c_1 + \cdots) + (a_2 + b_2 + c_2 + \ldots)]$$
$$T.A. = \tfrac{1}{2}s(p_1 + p_2) + B_1 + B_2$$
$$V = \tfrac{1}{3}h[B_1 + B_2 + \sqrt{B_1 + B_2}]$$

Frustum of Regular Pyramid



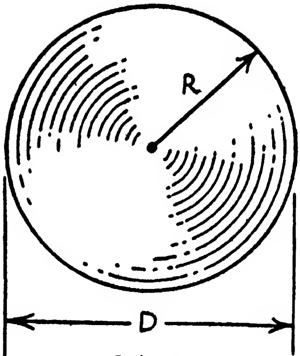

$$L.A. = \pi Rs$$
$$T.A. = \pi Rs + \pi R^2$$
$$= \pi R(s + R)$$
$$V = \tfrac{1}{3}\pi R^2 h$$
$$= \frac{\pi D^2 h}{12}$$
$$= .2618D^2h$$

Right Circular Cone



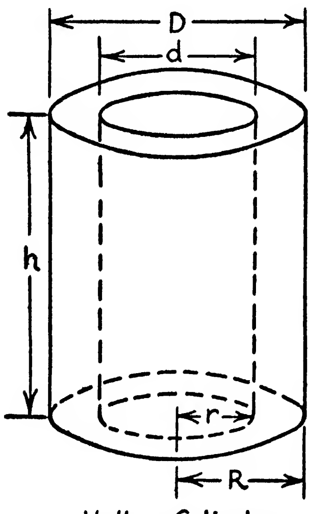

$$L.A. = \tfrac{1}{2}s(C + c)$$
$$= \tfrac{1}{2}s(2\pi R + 2\pi r)$$
$$= \pi s(R + r)$$
$$T.A. = \pi s(R + r) + \pi R^2 + \pi r^2$$
$$= \pi(R^2 + r^2 + Rs + rs)$$
$$V = \tfrac{1}{3}\pi h(R^2 + r^2 + Rr)$$
$$= .2618h(D^2 + d^2 + Dd)$$

Frustum of Right Circular Cone




Sphere

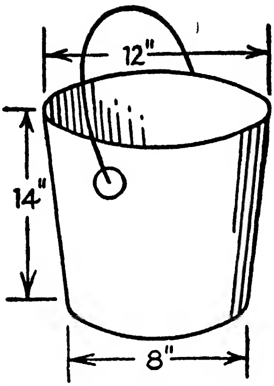
$$A=4\pi R^2=12.566R^2=\pi D^2$$
$$V=\tfrac{4}{3}\pi R^3=4.1888R^3$$
$$=\frac{\pi D^3}{6}=.5236D^3$$

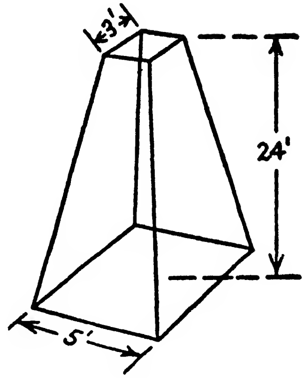



Hollow Cylinder

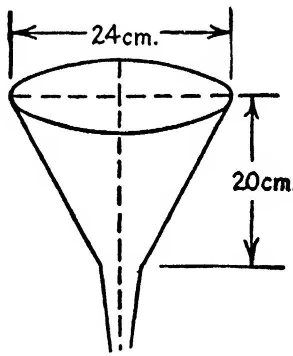
$$V=\pi h(R^2-r^2)$$
$$=\frac{\pi h}{4}(D^2-d^2)$$

*Exercise 67.*

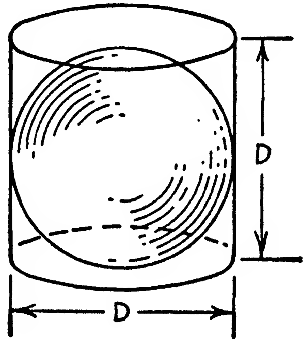



1. How many gallons can this bucket hold, if it has the form of a frustum of a right circular cone with dimensions as shown?
2. A concrete foundation pillar 24 ft. high is a frustum of a square pyramid whose bases are 3 ft. and 5 ft. on a side. How many cu. yd. of concrete are required for the pillar?




3. A funnel measures 24 cm. across the top, and 20 cm. from top to bottom, excluding the stem. Approximately how many cu. cm. of liquid will it hold when full?

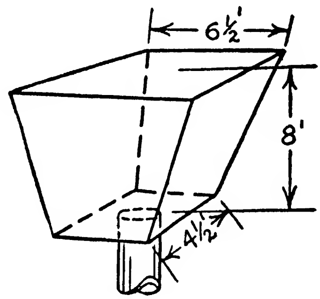



4. A sphere when immersed in a cylinder full of water just touches the sides and bottom, and is flush with the surface of the water remaining. What per cent of the water in the cylinder originally will the overflow be?

5. How many sq. in. of sheet metal will be needed to make a pail 12″ in diameter at the bottom and 14″ across at the top, with a slant height of 18″?

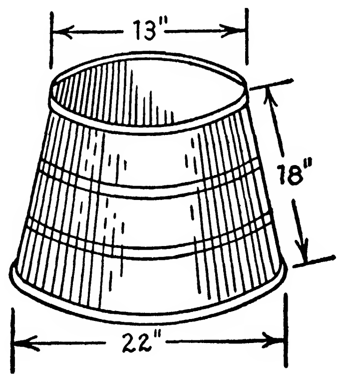
6. The altitude of a regular hexagonal pyramid is 17½″; the hexagonal base measures 4″ across the flats. Find the volume of the pyramid.

7. A feed hopper is in the form of the frustum of a square pyramid with dimensions shown. Find its cubic contents when filled to its maximum height of 8 ft.



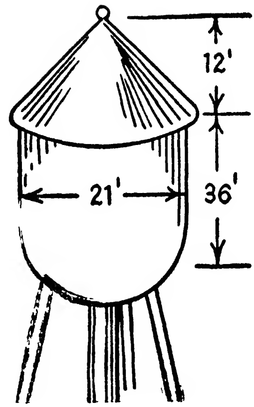

8. A conical pile of sand measures 66 ft. around the circumference of its base. If the slope of the pile is 45°, how many cu. yd. of sand are there in the pile?

9. A parchment lamp shade has the shape of the frustum of a circular cone with the dimensions shown; find the amount of material required to make it up.

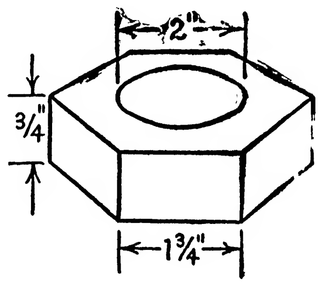



10. A cylindrical "capsule" with hemispherical ends has a diameter of 18″ and is 4½ feet in total length. Find its volume.

11. A regular triangular prism of glass measures 16.2 cm. in length; the side of each base is 1.8 cm. long. Find the lateral surface of the prism.
12. Find the volume of metal in a hollow sphere if the metal is 1½″ thick and the inside diameter equals 6″.
13. A cone pulley is a frustum of a cone with an altitude of 10½″; the diameters of the bases are 4″ and 6″, respectively. If a 1″ hole is drilled through the pulley, find the volume of the pulley.
14. Find the ratio between the surfaces of two spheres whose radii are 3″ and 6″.



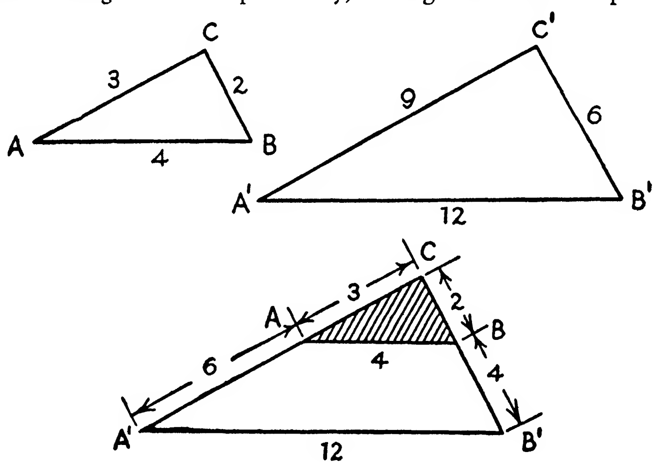

15. A water tank in the shape of a cylinder with a hemispherical bottom and a conical top has the dimensions shown. What is its maximum capacity in gallons?




16. A 2″ hole is drilled through a hexagonal piece of stock ¾″ thick and 1¾″ across the flats. Find the volume of metal in the finished piece.

## 19. SIMILAR FIGURES

**Similar Triangles.** It was previously pointed out that any two (or more) triangles having the same shape, although differing in size, were said to be *similar triangles*. More specifically, having the "same shape" means



that the corresponding angles of the two triangles are respectively equal, and that their corresponding sides are in proportion. Thus: $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$; also, $\frac{AC}{A'C'} = \frac{AB}{A'B'} = \frac{BC}{B'C'}$.

In other words, in this particular instance, any pair of corresponding sides of the two triangles has the ratio 3:1, (or 1:3); thus:

$$\frac{AC}{A'C'} = \frac{3}{9} = \frac{1}{3};\ \frac{AB}{A'B'} = \frac{4}{12} = \frac{1}{3};$$
$$\frac{B'C'}{BC} = \frac{6}{2} = 3:1;\ \frac{A'C'}{AC} = \frac{9}{3} = 3:1;\ \text{etc.}$$

As a matter of fact, *congruent* triangles may be regarded as a "special case" of similar triangles whose corresponding sides are in the ratio of 1:1.

**Conditions for Similarity.** Triangles are similar under any one of the following conditions:

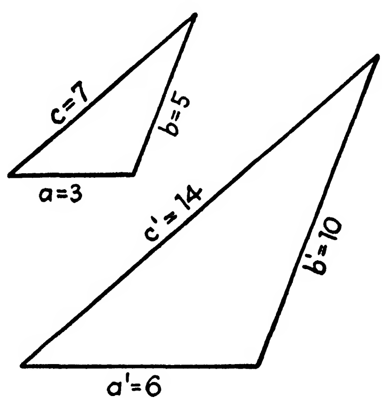
(1) if all three pairs of corresponding angles are equal;
(2) if any two pairs of corresponding angles are equal;
(3) if one pair of corresponding angles is equal, and the sides including them are in proportion;
(4) if all three pairs of corresponding sides are in proportion;
(5) if they are right triangles having one pair of acute angles equal.

**Proportional Parts.** As has just been seen, if two triangles are similar, their corresponding sides are in proportion; or

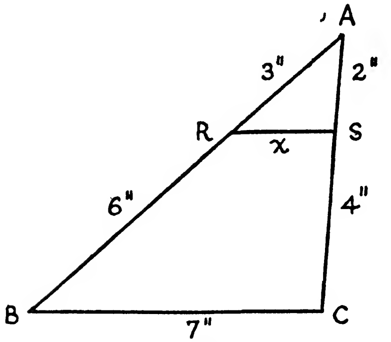
$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$
$$\frac{3}{6} = \frac{5}{10} = \frac{7}{14} = (\tfrac{1}{2}).$$

Or again; $\frac{a}{b} = \frac{a'}{b'};\ \frac{a}{c} = \frac{a'}{c'};\ \frac{b}{c} = \frac{b'}{c'}$

$$\frac{3}{5} = \frac{6}{10};\ \frac{3}{7} = \frac{6}{14};\ \frac{5}{7} = \frac{10}{14}.$$



Moreover, if a line (RS) is drawn parallel to any side (BC) of a triangle, it divides the other two sides into proportional parts, since the triangle cut off by the line is similar to the original triangle. Thus if RS divides AB in the ratio of 3:6 (or 1:2), it will also divide AC in the ratio of 1:2 (2:4). In other words,




$$\frac{AR}{RB}=\frac{AS}{SC}, \text{ or } 3/6=2/4.$$

Also, the length of RS can be determined if BC is known, for:

$$\frac{AR}{AB}=\frac{RS}{BC}, \text{ or } 3/9=\frac{x}{7};\ x=2\tfrac{1}{3}.$$

$$\text{Or again, } \frac{RS}{BC}=\frac{AS}{AC}, \text{ or } \frac{x}{7}=2/6;\ x=2\tfrac{1}{3}.$$

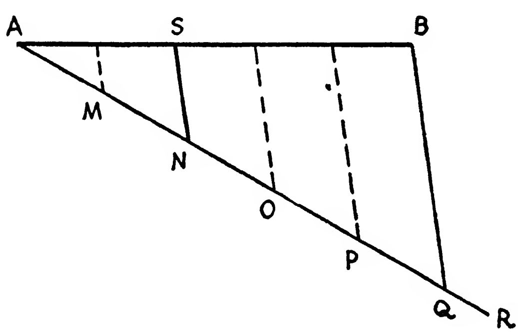
Furthermore, any segment is to the entire side as the corresponding segment is to the other side; i.e.,

$$\frac{AR}{AB}=\frac{AS}{SC}, \text{ or } \frac{3}{3+6}=\frac{2}{2+4};$$

$$\text{and } \frac{RB}{AB}=\frac{SC}{AC}, \text{ or } \frac{6}{3+6}=\frac{4}{2+4}.$$

**Dividing a Line into a Given Ratio.** A simple method for dividing a given line into any number of parts having any desired ratio is the following construction, based on similar triangles.

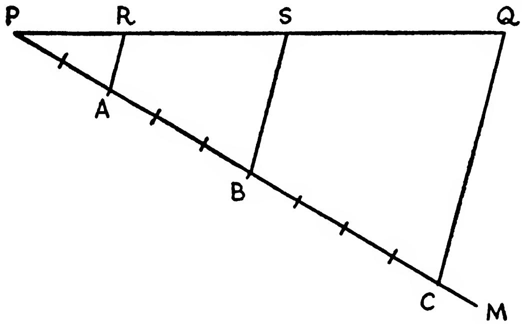
(a) Suppose that the line AB is to be divided into two parts having the ratio 2:3. At any convenient angle, draw an indefinite line through A. On AR step off five convenient equal segments, AM, MN, NO, OP, and PQ. Join Q with B, and through N construct NS || QB. The two required segments are AS and SB.



For, by the construction, $\triangle$ANS is similar to ($\sim$) $\triangle$AQB; therefore

$$\frac{AN}{NQ}=\frac{AS}{SB}=2/3.$$

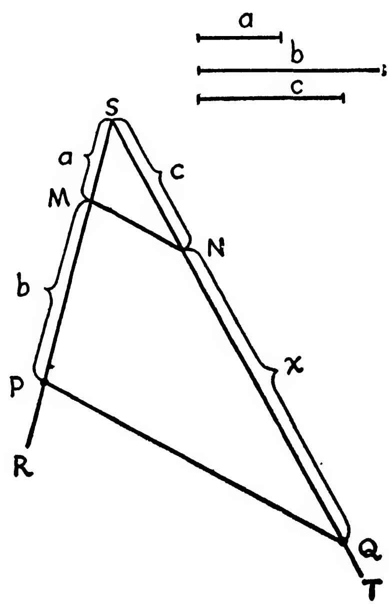
(b) Suppose that a given line PQ is to be divided into three parts having the ratio 2:3:4; a similar procedure is followed. An indefinite line PM is drawn at any convenient angle to PQ; 9 equal segments are then stepped off on PM, beginning at P and ending at C. Point C is joined with Q, and lines constructed through A and B, respectively, parallel to QC, and intersecting PQ in R and S. The required segments are PR, RS and SQ; for



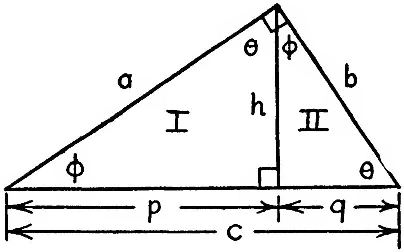

$$PA:AB:BC=PR:RS:SQ=2:3:4$$

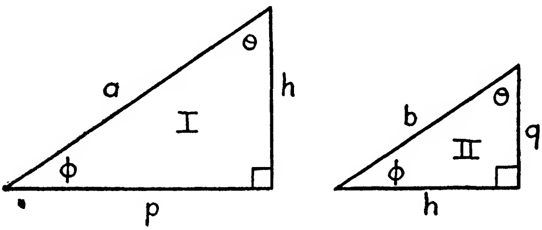
**Constructing a Fourth Proportional.** The same principle may be used to construct a fourth proportional to three given lines; for example, find the fourth proportional ($x$) to the given segments $a$, $b$ and $c$. On any convenient angle RST with indefinite sides, lay off the given segments $a$, $b$ and $c$ as shown. Join the ends M and N of segments $a$ and $c$; through P, the end of segment $b$, construct a line parallel to MN, intersecting ST in Q. Then NQ is the required fourth proportional $x$; for

$$\frac{SM}{MP}=\frac{SN}{NQ}, \text{ or } \frac{a}{b}=\frac{c}{x}.$$



**The Altitude upon the Hypotenuse.** A little consideration will show that if the altitude of a right triangle is drawn to the hypotenuse, the two triangles thus formed are not only similar to each other, but each is similar to the original triangle as well. As a result, the following relations hold; they are occasionally useful in computations and in deriving other properties of such figures:



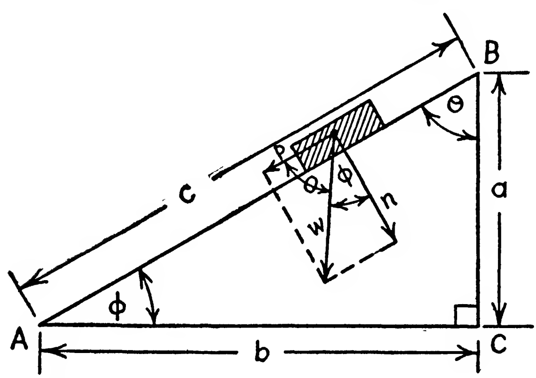




(1) $\dfrac{h}{p}=\dfrac{q}{h}$, or $h^2=pq,\ h=\sqrt{pq}.$

(2) $\dfrac{a}{p}=\dfrac{p+q}{a}$, or $a^2=p\ (p+q)$;
$a^2=pc$, or $a=\sqrt{pc}.$

(3) $\dfrac{b}{q}=\dfrac{p+q}{b}$, or $b^2=q\ (p+q)$;
$b^2=qc$, or $b=\sqrt{qc}.$

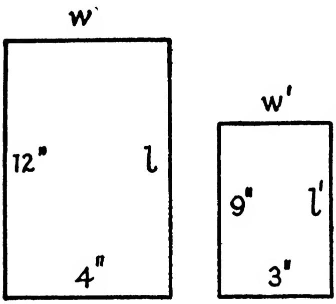
**Practical Problem.** A useful application of similar triangles occurs in connection with a study of forces acting on a body on an inclined plane. Thus the triangle of which $w$ and $p$ (the downward pull of the weight, and the pull along the inclined plane, respectively)are two sides of a triangle is similar to △ ABC; so is the triangle having $w$ and $n$ for two of its sides, where $n$ is the "normal" (i.e., perpendicular) push upon the plane. Hence:



$$\frac{p}{w}=\frac{a}{c}, \text{ or } p=\left(\frac{a}{c}\right)w;\quad \frac{n}{w}=\frac{b}{c}, \text{ or } n=\left(\frac{b}{c}\right)w;\quad \frac{p}{n}=\frac{a}{b}.$$

In other words, if $a:b$, for example, equals 1:2, then the pull down the plane is half as great as the pressure exerted perpendicularly against the plane; and so on.

**Similar Rectangles.** Triangles, of course, are not the only geometric figures which may be similar in shape. Thus if the corresponding sides of two rectangles are in proportion, the rectangles are said to be similar. Here




$$\frac{w}{l}=\frac{w'}{l'}, \text{ or } \frac{w}{w'}=\frac{l}{l'}.$$

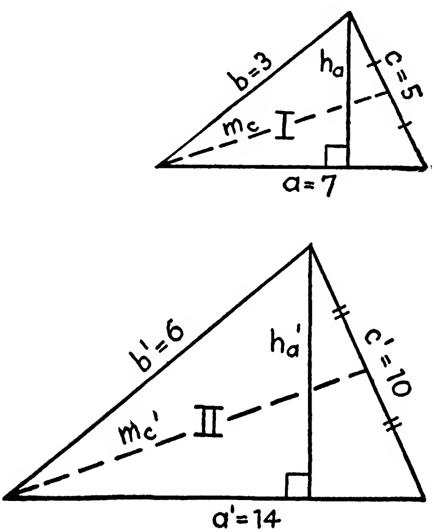
Moreover, the perimeters of two similar rectangles are in the same ratio as any two corresponding sides; i.e.,

$$\frac{P}{P'}=\frac{l}{l'}=\frac{w}{w'}, \text{ or}$$

$$32/24=12/9=4/3.$$

It should also be noted that any two squares are "automatically" similar, and that therefore their perimeters are also in the same ratio as the corresponding sides of the squares.

**General Properties of Similar Triangles.** This property of the perimeters of similar rectangles holds true of similar triangles also; in fact, in two similar triangles, *any two corresponding "parts"*—altitudes, medians, angle-bisectors—have the same ratio as a pair of corresponding sides; thus in similar triangles I and II we have:

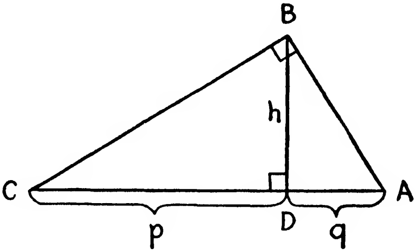



$$\frac{P}{P'}=\frac{ha}{ha'}=\frac{mc}{mc'}=\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}=1/2;$$

the same ratio (1:2) would likewise hold for any other pair of corresponding altitudes or corresponding medians.

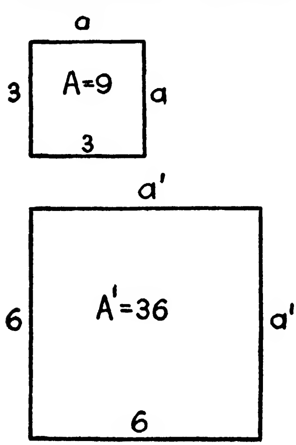
### *Exercise 68.*

1. Two similar rectangles have respective bases of 6″ and 15″. If the altitude of the first is 10″, what is the perimeter of the second rectangle?
2. The base of a triangle is 6 ft. and its altitude is 4 ft. If the corresponding base of a similar triangle is 18 ft., what is the area of the second triangle?
3. If AD=8 and AC=18, find the length of AB; the length of BD.
4. If $h=12$ and $q=9$, find CA and CB.



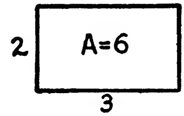

5. A tree casts a shadow 90 ft. long at the same time that a nearby post 6 ft. high casts a shadow 4 ft. long. How high is the tree?
6. The base of a triangle is 20″, and the other sides are 10″ and 16″. A line parallel to the base cuts off 2″ from the lower end of the shorter side. Find (a) the segments of the other side, and (b) the length of the parallel line.
7. The sides of a right triangle are 9″, 12″ and 15″. Find (a) the altitude to the hypotenuse, and (b) the segments of the hypotenuse into which this altitude divides it.
8. The bases of a trapezoid are 6″ and 8″, and the other two sides are 4″ and 5″ respectively. How far must each of these latter sides be produced before they meet in a point?

**Areas of Similar Figures.** It will be noted that in the two squares here shown, the ratio of the sides is 1:2, while the ratio of their *areas* is 1:4; or, doubling the side multiplies the area by *four,* and *not* by two. Similarly, tripling the sides of a rectangle multiplies the area by 9 instead of by 3. In short, the *areas of any two similar figures* are to each other as the *squares* of their respective sides, or the squares of any of their corresponding dimensions. Thus,

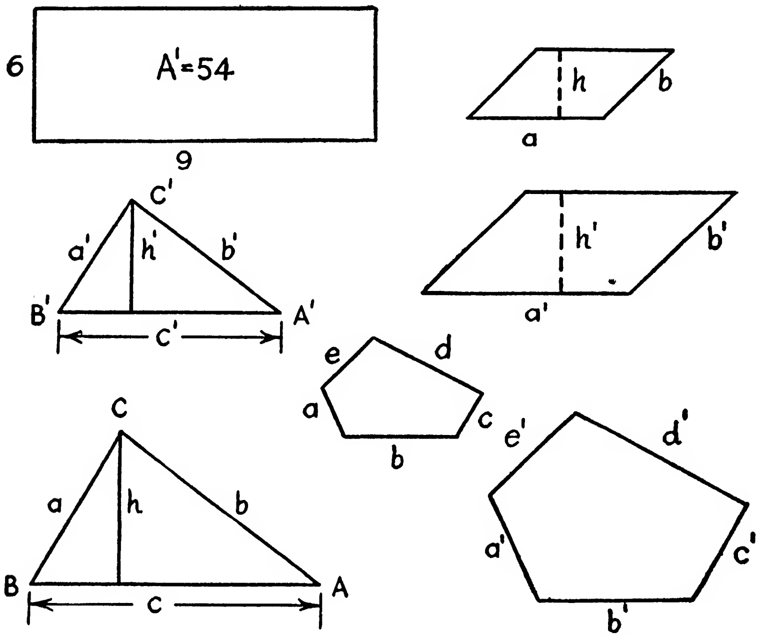



$$\frac{\text{Area}\triangle\ ABC}{\text{Area}\triangle A'B'C'}=\frac{a^2}{a'^2}=\frac{b^2}{b'^2}=\frac{c^2}{c'^2}=\frac{h^2}{h'^2}=\frac{P^2}{P'^2},$$

where $P$ and $P'$ represent their perimeters. So also in the case of all similar polygons, i.e., polygons having their respective angles

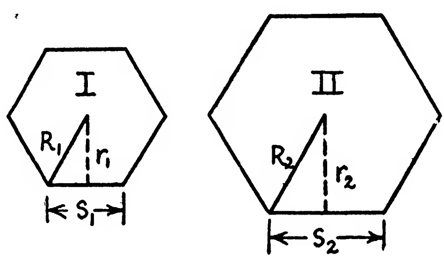



equal and their corresponding sides proportional: the areas of such polygons are to each other as the squares of their corresponding sides, altitudes, etc.




**Regular Polygons; Circles.** It is easily seen that if any two *regular* polygons have the same number of sides they must be similar. The *perimeters* of any two such regular polygons are to each other as any two corresponding sides; so also is the ratio of their corresponding radii and apothems. Expressed symbolically, we have:

$\frac{P_{I}}{P_{II}} = \frac{s_1}{s_2} = \frac{R_1}{R_2} = \frac{r_1}{r_2}$, where $P_{I}$ and $P_{II}$ denote their perimeters, $R_1$ and $R_2$ their radii, $r_1$ and $r_2$ their apothems, and $s_1$ and $s_2$ their sides, respectively.

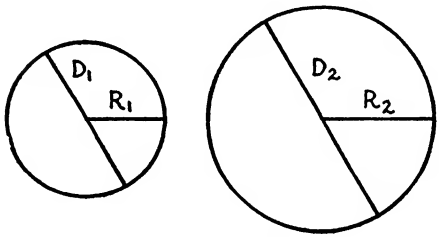



The *areas* of two regular polygons having the same number of sides are in the same ratio as the *squares* of their corresponding parts: thus

$$\frac{A_I}{A_{II}}=\frac{s_1^2}{s_2^2}=\frac{R_1^2}{R_2^2}=\frac{r_1^2}{r_2^2}=\frac{P_I^2}{P_{II}^2}.$$

In the case of circles the relations are very much the same; in a sense, any two circles may be regarded as "similar figures." The perimeters (i.e., circumferences) of any two circles are to each other as their respective radii or diameters; their areas, however, are to each other as the squares of their corresponding radii, diameters, or circumferences; or, $\frac{C_1}{C_2}=\frac{R_1}{R_2}=\frac{D_1}{D_2}$, and

$$\frac{A_1}{A_2}=\frac{R_1^2}{R_2^2}=\frac{D_1^2}{D_2^2}=\frac{C_1^2}{C_2^2}.$$




EXAMPLE 1: The distances across the flats of two hexagonal plates are 1″ and 1½″, respectively. What is the ratio of their cross-sectional areas?

SOLUTION: $\frac{A_1}{A_2}=\frac{(1)^2}{(1\frac{1}{2})^2}=\frac{\frac{1}{9}}{4}=\frac{4}{9}$, *Ans.*

EXAMPLE 2: Two circular discs are 3″ and 5″ in diameter, respectively. How many times larger than the first disc is the second?

SOLUTION: $\frac{A_2}{A_1}=\frac{5^2}{3^2}=\frac{25}{9}=2\frac{7}{9}$ times as large, *Ans.*

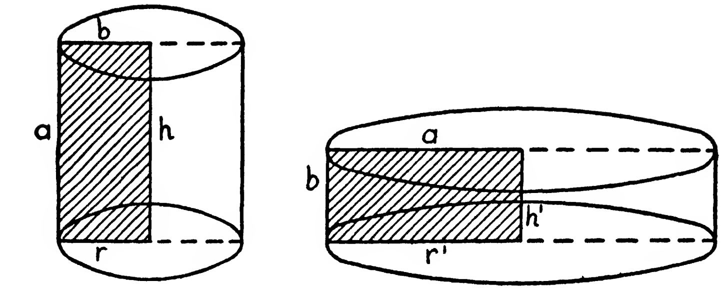
EXAMPLE 3: The area of one circular plate is twice that of another. Find the ratio of their circumferences.

SOLUTION: $\frac{A_1}{A_2}=\frac{2}{1}=\frac{R_1^2}{R_2^2}=\left(\frac{R_1}{R_2}\right)^2$

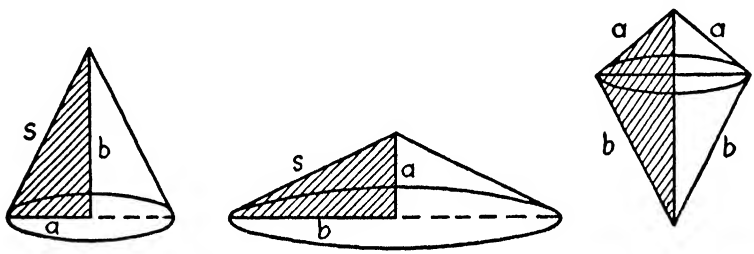
$$\left(\frac{R_1}{R_2}\right)=\sqrt{\tfrac{2}{1}}=\sqrt{2}=1.41$$

$$\frac{C_1}{C_2}=\frac{R_1}{R_2}=1.41, \textit{ Ans.}$$

**Similar Solids of Revolution.** A right circular cylinder may be regarded as having been formed by revolving a rectangle about either of its sides as an axis of rotation; the side of the rectangle about which it is revolved becomes the altitude of the cylinder, and the other side becomes the radius of the base. In the same manner, a right circular cone can be regarded as the solid that is formed by revolving a right triangle about

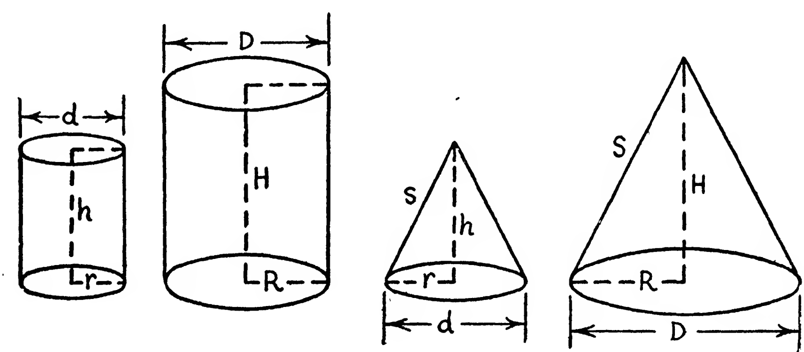



either of its sides as an axis; the side around which it is revolved becomes the altitude, the other side becomes the radius of the base, and the hypotenuse becomes the slant height. If a right triangle is revolved around




the hypotenuse as an axis, two circular cones are formed (one inverted), having a common base, and a *combined altitude* equal to the hypotenuse.

If now we consider cylinders and cones of revolution formed from similar rectangles and similar right triangles, respectively, we have *similar solids of revolution;* their *areas,* lateral and total, are to each other as the *squares* of their corresponding dimensions, and their *volumes* are to each other as the *cubes* of their corresponding dimensions.




I II I II

Thus for similar cylinders of revolution:

$$\frac{\text{L.A.}_{\text{I}}}{\text{L.A.}_{\text{II}}}=\frac{\text{T.A.}_{\text{I}}}{\text{T.A.}_{\text{II}}}=\frac{h^2}{H^2}=\frac{r^2}{R^2}=\frac{d^2}{D^2};$$

$$\frac{\text{V}_{\text{I}}}{\text{V}_{\text{II}}}=\frac{h^3}{H^3}=\frac{r^3}{R^3}=\frac{d^3}{D^3}.$$

And for similar cones of revolution:

$$\frac{\text{L.A.}_{\text{I}}}{\text{L.A.}_{\text{II}}}=\frac{\text{T.A.}_{\text{I}}}{\text{T.A.}_{\text{II}}}=\frac{h^2}{H^2}=\frac{r^2}{R^2}=\frac{d^2}{D^2}=\frac{s^2}{S^2};$$

$$\frac{\text{V}_{\text{I}}}{\text{V}_{\text{II}}}=\frac{h^3}{H^3}=\frac{r^3}{R^3}=\frac{d^3}{D^3}=\frac{s^3}{S^3}.$$

**Spheres.** Just as any two circles may be regarded as similar, so any two spheres may be regarded as similar. Therefore we have

$$\frac{\text{A}_{\text{I}}}{\text{A}_{\text{II}}}=\frac{r^2}{R^2}=\frac{d^2}{D^2}, \text{ and } \frac{\text{V}_{\text{I}}}{\text{V}_{\text{II}}}=\frac{r^3}{R^3}=\frac{d^3}{D^3}.$$

EXAMPLE 1: Two cylindrical tin cans are similar, i.e., their heights and diameters are in the proportion of 2:3. Find (a) the ratio of their lateral surfaces; (b) the ratio of their volumes.

SOLUTION: $\frac{\text{L.A.}_{\text{I}}}{\text{L.A.}_{\text{II}}}=\frac{2^2}{3^2}=4/9$, *Ans.*

$\frac{\text{V}_{\text{I}}}{\text{V}_{\text{II}}}=\frac{2^3}{3^3}=8/27$, *Ans.*

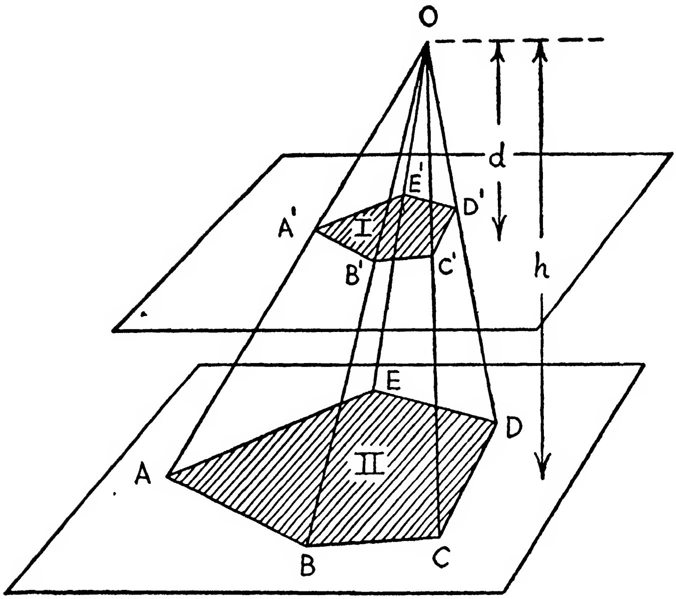
EXAMPLE 2: A right circular cone has a diameter of 4″ and an altitude of 5″. Another cone, similar to the first, has an altitude of 10″. Find (a) the diameter of the second cone; (b) the ratio of their total areas; (c) how many times larger in volume the second cone is than the first.

SOLUTION: (a) $4/5=\frac{x}{10}$; $5x=40$; $x=8$, *Ans.*

(b) $\frac{\text{T.A.}_{\text{I}}}{\text{T.A.}_{\text{II}}}=\frac{4^2}{5^2}=16/25$, *Ans.*

(c) $\frac{\text{V}_{\text{II}}}{\text{V}_{\text{I}}}=\frac{5^3}{4^3}=125/64=1\,61/64$ times as large, *Ans.*

**Section of Pyramid Parallel to the Base.** If in any pyramid, whether a regular pyramid or an oblique pyramid, a plane section is passed parallel to the base, the polygon formed is similar to the base, and the pyramid




"cut off" is similar to the original pyramid. Furthermore, the lateral edges will be divided in the ratio of $d:(h-d)$, where $d$ is the distance of the cutting plane below the vertex, and $h$ is the original altitude. Or:

$$\frac{OA'}{OA}=\frac{OB'}{OB}=\frac{OC'}{OC}\text{(etc.)}=\frac{d}{h};$$

also, $$\frac{A'B'}{AB}=\frac{B'C'}{BC}=\frac{C'D'}{CD}=\text{(etc.)}=\frac{d}{h}; \text{ and}$$

$$\frac{\text{area I}}{\text{area II}}=\frac{(OA')^2}{(OA)^2}=\frac{(A'B')^2}{(AB)^2}=\frac{d^2}{h^2}.$$

*Exercise 69.*

1. If the radius of a circle is cut in half, how is its circumference changed? its area?
2. The areas of two circles are as 4:9; if the radius of the larger circle is 12 inches, what is the circumference of the smaller circle?
3. If the edge of a cube is doubled, how is its area changed? its volume?
4. The diameters of two circles are 6″ and 8″ respectively. Find (a) the ratio of their circumference, and (b) the ratio of their areas.
5. The circumference of a circle is 20″. What is the circumference of a circle having twice the area of the given circle?

6. If the diameter of a cylindrical shaft is diminished by 10%, how is its circumference changed? the area of its cross section?
7. A water storage tank is fed by two pipe lines, one 8″ and one 6″ in diameter. If these are to be replaced by a single pipe line having the same capacity as the two combined, what diameter pipe should be used? (*Hint:* $\pi R^2+\pi r^2=\pi x^2$.)
8. A rectangular zinc cut for photoengraving is reduced "three-fourths"; what is the ratio of the areas involved?
9. If the diameter of a water main is made half again as large, what is the per cent of increase in the capacity of the pipe? (*Hint:* compare the cross-sectional areas.)
10. Find the ratio of the volume of two spheres if their areas are in the ratio of 3:1.
11. Find the lateral area and the volume of a cone of revolution if the radius of the base is 8″ and the slant height forms an angle of 60° with the plane of the base.
12. A cylindrical container 8″ high has a diameter of 4″. If each dimension is increased by 25%, what is the ratio of the new total surface to the original total surface? What is the per cent of increase in total surface?
13. An equilateral triangle whose side is 6″ is revolved about one of its sides as an axis. Find the total surface and volume of the solid generated.
14. The area of the base of a circular cone is 108 sq. in., and its altitude is 6″. If a section is passed parallel to the base and 4″ above it, what is the area of the base of the cone cut off?
15. If a cylindrical metal drum used for shipping chemicals has each dimension increased by 20%, what is the per cent of increase in its capacity?
16. If the inside diameter of a pipe is increased 50%, how much more water will flow through it at the same rate in the same amount of time?
17. If the height of a cylinder is cut in half, and the diameter is doubled, what is the change in volume?
18. A micro-photograph was enlarged "20 diameters." How many times larger than the original is the area of the photograph?
19. One of two circular water pipes is 4″ in diameter, and the other is 8″. (a) How many times larger is the cross-sectional area of the second? (b) How much more water will flow through the second pipe?
20. Because of cloudy weather a photographer enlarged the diaphragm (circular opening) of his camera from a diameter of 0.4 cm. to 0.6 cm. By what per cent did this increase the area of the opening?

## TABLE OF TANGENTS, COSINES, AND SINES

| Angle | Tangent (opp./adj.) | Cosine (adj./hyp.) | Sine (opp./hyp.) | Angle | Tangent (opp./adj.) | Cosine (adj./hyp.) | Sine (opp./hyp.) |
|---|---|---|---|---|---|---|---|
| 0° | .0000 | 1.0000 | .0000 | 45° | 1.0000 | .7071 | .7071 |
| 1° | .0175 | .9998 | .0175 | 46° | 1.0355 | .6947 | .7193 |
| 2° | .0349 | .9994 | .0349 | 47° | 1.0724 | .6820 | .7314 |
| 3° | .0524 | .9986 | .0523 | 48° | 1.1106 | .6691 | .7431 |
| 4° | .0699 | .9976 | .0698 | 49° | 1.1504 | .6561 | .7547 |
| 5° | .0875 | .9962 | .0872 | 50° | 1.1918 | .6428 | .7660 |
| 6° | .1051 | .9945 | .1045 | 51° | 1.2349 | .6293 | .7771 |
| 7° | .1228 | .9925 | .1219 | 52° | 1.2799 | .6157 | .7880 |
| 8° | .1405 | .9903 | .1392 | 53° | 1.3270 | .6018 | .7986 |
| 9° | .1584 | .9877 | .1564 | 54° | 1.3764 | .5878 | .8090 |
| 10° | .1763 | .9848 | .1736 | 55° | 1.4281 | .5736 | .8192 |
| 11° | .1944 | .9816 | .1908 | 56° | 1.4826 | .5592 | .8290 |
| 12° | .2126 | .9781 | .2079 | 57° | 1.5399 | .5446 | .8387 |
| 13° | .2309 | .9744 | .2250 | 58° | 1.6003 | .5299 | .8480 |
| 14° | .2493 | .9703 | .2419 | 59° | 1.6643 | .5150 | .8572 |
| 15° | .2679 | .9659 | .2588 | 60° | 1.7321 | .5000 | .8660 |
| 16° | .2867 | .9613 | .2756 | 61° | 1.8040 | .4848 | .8746 |
| 17° | .3057 | .9563 | .2924 | 62° | 1.8807 | .4695 | .8829 |
| 18° | .3249 | .9511 | .3090 | 63° | 1.9626 | .4540 | .8910 |
| 19° | .3443 | .9455 | .3256 | 64° | 2.0503 | .4384 | .8988 |
| 20° | .3640 | .9397 | .3420 | 65° | 2.1445 | .4226 | .9063 |
| 21° | .3839 | .9336 | .3584 | 66° | 2.2460 | .4067 | .9135 |
| 22° | .4040 | .9272 | .3746 | 67° | 2.3559 | .3907 | .9205 |
| 23° | .4245 | .9205 | .3907 | 68° | 2.4751 | .3746 | .9272 |
| 24° | .4452 | .9135 | .4067 | 69° | 2.6051 | .3584 | .9336 |
| 25° | .4663 | .9063 | .4226 | 70° | 2.7475 | .3420 | .9397 |
| 26° | .4877 | .8988 | .4384 | 71° | 2.9042 | .3256 | .9455 |
| 27° | .5095 | .8910 | .4540 | 72° | 3.0777 | .3090 | .9511 |
| 28° | .5317 | .8829 | .4695 | 73° | 3.2709 | .2924 | .9563 |
| 29° | .5543 | .8746 | .4848 | 74° | 3.4874 | .2756 | .9613 |
| 30° | .5774 | .8660 | .5000 | 75° | 3.7321 | .2588 | .9659 |
| 31° | .6009 | .8572 | .5150 | 76° | 4.0108 | .2419 | .9703 |
| 32° | .6249 | .8480 | .5299 | 77° | 4.3315 | .2250 | .9744 |
| 33° | .6494 | .8387 | .5446 | 78° | 4.7046 | .2079 | .9781 |
| 34° | .6745 | .8290 | .5592 | 79° | 5.1446 | .1908 | .9816 |
| 35° | .7002 | .8192 | .5736 | 80° | 5.6713 | .1736 | .9848 |
| 36° | .7265 | .8090 | .5878 | 81° | 6.3138 | .1564 | .9877 |
| 37° | .7536 | .7986 | .6018 | 82° | 7.1154 | .1392 | .9903 |
| 38° | .7813 | .7880 | .6157 | 83° | 8.1443 | .1219 | .9925 |
| 39° | .8098 | .7771 | .6293 | 84° | 9.5144 | .1045 | .9945 |
| 40° | .8391 | .7660 | .6428 | 85° | 11.4301 | .0872 | .9962 |
| 41° | .8693 | .7547 | .6561 | 86° | 14.3007 | .0698 | .9976 |
| 42° | .9004 | .7431 | .6691 | 87° | 19.0811 | .0523 | .9986 |
| 43° | .9325 | .7314 | .6820 | 88° | 28.6363 | .0349 | .9994 |
| 44° | .9657 | .7193 | .6947 | 89° | 57.2900 | .0175 | .9998 |
| 45° | 1.0000 | .7071 | .7071 | 90° |  | .0000 | 1.0000 |

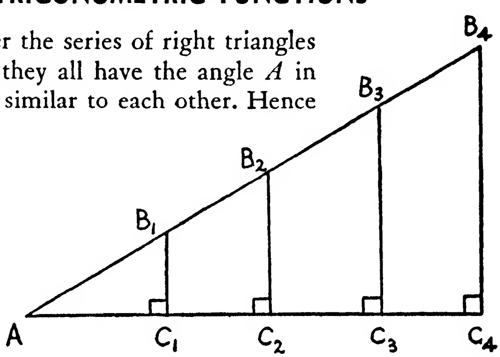
# CHAPTER IV

# SHOP TRIGONOMETRY

*William L. Schaaf*

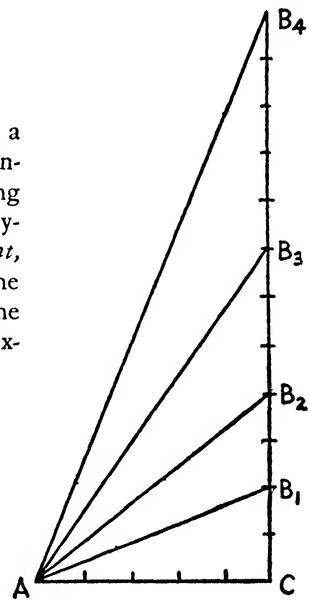
## 20. USING TRIGONOMETRIC FUNCTIONS

**Similar Triangles.** Consider the series of right triangles $AB_1C_1$, $AB_2C_2$, etc. Since they all have the angle $A$ in common, the triangles are similar to each other. Hence the ratios of the corresponding pairs of sides are equal; or




$$\frac{B_1C_1}{AC_1}=\frac{B_2C_2}{AC_2}=\frac{B_3C_3}{AC_3}=\frac{B_4C_4}{AC_4}=k,$$

where $k$ represents some numerically constant value, independent of the units of measure used. In other words, the value of $k$ could be used as a *measure of angle A*. Furthermore, if we consider another series of right triangles having one side fixed, but with the angle at A varying, then the series of ratios is *not constant*, but varies with the size of the angle *A;* if the angle increases, the ratio increases, and if the angle decreases, the ratio decreases. For example:




$$\frac{B_1C}{AC}=\frac{2}{5}=.4;\quad \frac{B_3C}{AC}=\frac{7}{5}=1.4;$$

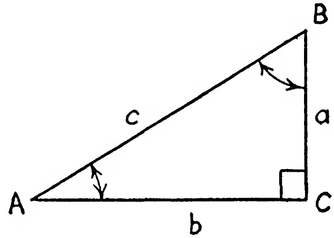
$$\frac{B_2C}{AC}=\frac{4}{5}=.8;\quad \frac{B_4C}{AC}=\frac{12}{5}=2.4;\text{ etc.}$$

In any right triangle, the ratio of the side opposite an acute angle to the side adjacent to it is called *the tangent of that angle;* this is written:

$$\tan A = \frac{BC}{AC} = \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}}.$$

Similarly:

$$\tan B = \frac{AC}{BC} = \frac{b}{a}.$$




**Trigonometric Ratios.** In the same way, any other pair of sides of the right triangle might be used to form a ratio which can be regarded as a measure of the angle. Such ratios are called *trigonometric functions;* their names and definitions are given below:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c} \qquad \sec A = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{c}{b}$$

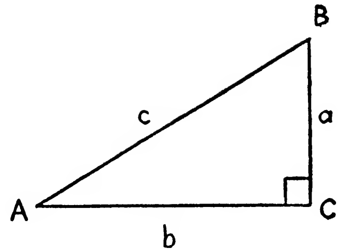
$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c} \qquad \csc A = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{c}{a}$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b} \qquad \cot A = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{b}{a}$$

It will be sufficient for most practical purposes to use the three functions in the first column, viz., *sine, cosine* and *tangent;* these should by all means be memorized. The other three, the *secant, cosecant,* and *cotangent,* are perhaps not so important for ordinary use.

**Table of Natural Functions.** The numerical values of these ratios for various angles from 1° to 90° have been carefully worked out to several decimal places; part of such a table is given above for the three most commonly used functions—the sine, the cosine, and the tangent. How such a table of the values of the various functions is used will be shown as we go along.

**Co-functions.** By studying the diagram and referring to the definitions once more, the following relations will be seen to hold:




$$\sin A = \frac{a}{c} = \cos B,$$

$$\text{and } \cos A = \frac{b}{c} = \sin B.$$

In other words, since $\angle A + \angle B = 90°$, the sine of any angle equals the cosine of its complement, and the cosine of an angle equals the sine of its complement. This explains the meaning of the word cosine, i.e., *co-sine,* or "complement's sine." Thus if sin 32° = .5299, then cos 58° also equals .5299; etc. Similarly,

$$\tan A = \frac{a}{b} = \cot B;$$

$$\cot A = \frac{b}{a} = \tan B;$$

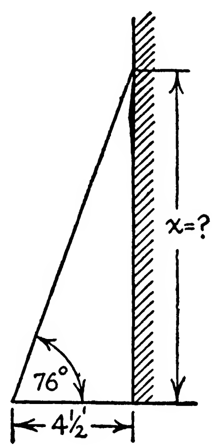
and also:

$$\sec A = \frac{c}{b} = \csc B;$$

$$\csc A = \frac{c}{a} = \sec B.$$

**Using the Tangent.** Several common types of problems arise where the use of the tangent is very convenient.

EXAMPLE 1: A ladder leaning against the wall of a building makes an angle of 76° with the ground. If the foot of the ladder is 4½ ft. from the base of the wall, how high above the ground is the point on the wall where the top of the ladder touches it?




SOLUTION:

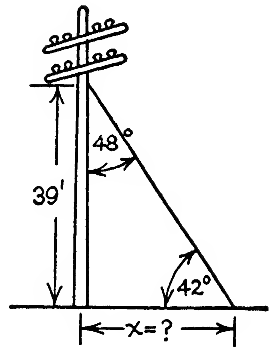
$$\tan 76° = \frac{x}{4.5} \quad \text{(from the figure)}$$

$$\tan 76° = 4.0108 \quad \text{(from the table)}$$

$$\text{hence } \frac{x}{4.5} = 4.0108$$

$$\text{or } x = (4.5)(4.0108) = 18.05 \text{ ft., } \textit{Ans.}$$

EXAMPLE 2: The guy wire supporting a telegraph pole is fastened to the pole at a point 39 ft. above the ground, and makes an angle of 42° with the ground. Find how far from the base of the pole must the stake be driven to fasten the guy wire.

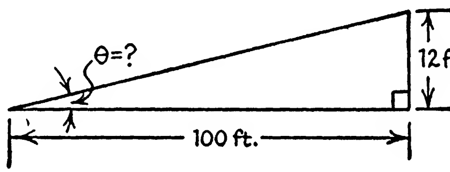



SOLUTION:

$$\frac{x}{39}=\tan 48°=1.1106$$

$x=(39)(1.1106)=43.3$ ft., *Ans.*

EXAMPLE 3: A ramp rises 12 ft. in a horizontal distance of 100 ft. Find the angle ($\theta$) of inclination.




SOLUTION:

tan $\theta=12/100=.1200$

From the table: tan $7°=.1228$

$\theta=7°$ (approx.), *Ans.*

NOTE: More exact results can be found by interpolation similar to that used in finding logarithms, or by using a more complete table of trigonometric functions showing values for degrees and minutes. Such a table is given at the back of this book. Thus, by using the complete table, the angle in Ex. 3, expressed to the nearest minute, is 6° 51′.

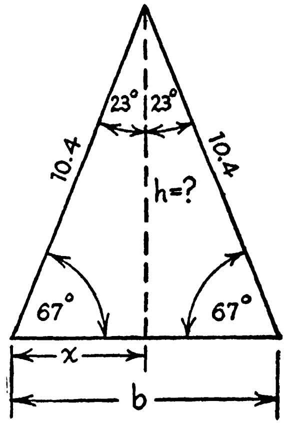
*Exercise 70.*

In the following problems, use the table of values at the back of the book when finding angles, obtaining your result to the nearest minute.

1. Find the side of an equilateral triangle whose altitude is 22.8″.
2. The slope of a roof is 4 inches in each horizontal foot. What angle does it make with the horizontal?
3. The altitude of an isosceles triangle is 12 and the base is 4. Find the length of the equal sides and the angles of the triangle.
4. A railroad track makes an angle of 10° with the horizontal. How many feet does it rise in 1000 ft. along the horizontal?
5. Find the apothem of a regular octagon whose sides are 4″ each in length.
6. A rectangle is 36″×48″. Find the angle between the diagonal and the longer side.

**Using Sines and Cosines.** How these functions are similarly utilized in practical problems involving right triangles will now be illustrated.

Example 1: Find the altitude of an isosceles triangle whose vertex angle is 46° and whose equal sides are each 10.4 inches. What is the length of its base?




Solution: $\dfrac{h}{10.4}=\sin 67°=.9205$

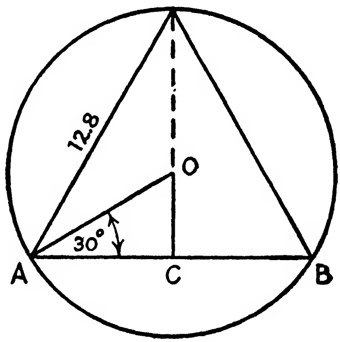
$h=(10.4)(.9205)=9.57$, *Ans.*

$\dfrac{x}{10.4}=\cos 67°=.3907$

$x=(10.4)(.3907)=4.06$

$b=(2)(4.06)=8.12$, *Ans.*

Example 2: Find the radius of the circle circumscribed around an equilateral triangle 12.8″ on a side.



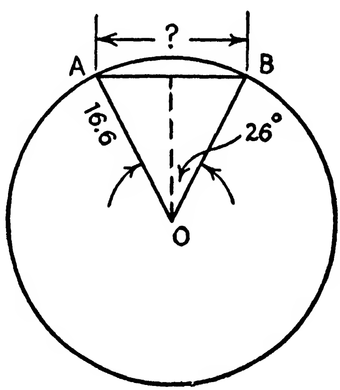

Solution:

$AC=\frac{1}{2}(12.8)=6.4$

$\dfrac{AC}{r}=\cos 30°$

$r=\dfrac{AC}{\cos 30°}=\dfrac{6.4}{.8660}=7.39$, *Ans.*

Example 3: How long is a chord of a circle that subtends an angle of 26° at the center, if the radius of the circle is 16.6 inches? What is the distance of the chord from the center?




Solution: $\dfrac{AM}{16.6}=\sin 13°=.2250$

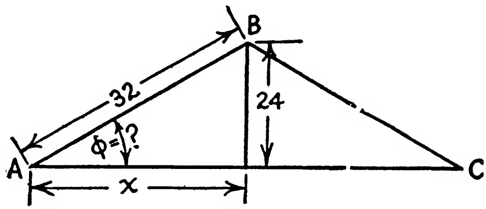
$AM=(16.6)(.2250)=3.74$

$\text{chord}=2\times(AM)=7.48$, *Ans.*

$\dfrac{OM}{16.6}=\cos 13°=.9744$

$OM=(16.6)(.9744)=16.18$, *Ans.*

**Example 4:** The equal sides of an isosceles triangle are each 32″ and the altitude is 24″. Find the base and the angles.




**Solution:**

$$\sin \phi=\frac{24}{32}=.7500$$

$$\phi=48°35', \textit{Ans.}$$

$$\frac{x}{32}=\cos 48°35'=.6615$$

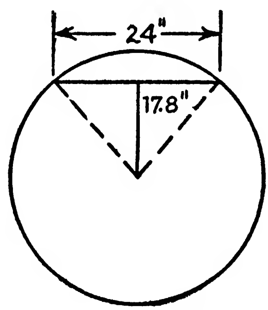
$$x=(32)(.6615)=21.17$$

$$AC=2x=(2)(21.17)=42.34, \textit{Ans.}$$

*Exercise 71.*

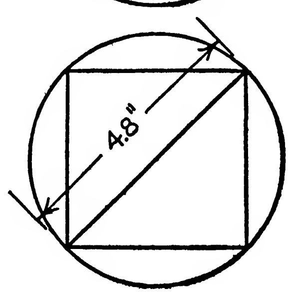
In the following problems, use the table in the back of the book when finding angles, obtaining the answer to the nearest minute.

1. Find the radius of a circle if a 12-inch chord subtends an angle of 28° at the center.




2. The equal sides of an isosceles triangle are 22″ long and the altitude is 9.2″. Find the base and the angles.

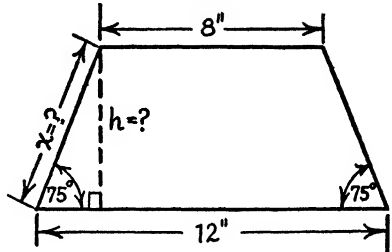
3. A chord 24 inches long is 17.8 inches distant from the center. Find (a) the radius of the circle, and (b) the subtended angle of the chord.




4. What is the largest square that can be milled from a circular disc 4.8 inches in diameter?

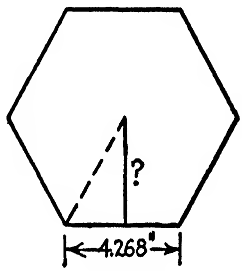
5. An isosceles triangle has sides of 16″, 16″ and 10″. Find the angles and the altitude.

6. The bases of an isosceles trapezoid are 8″ and 12″. If the base angles are 75°, find the equal sides and the altitude.

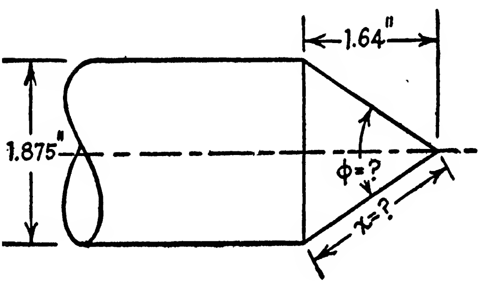


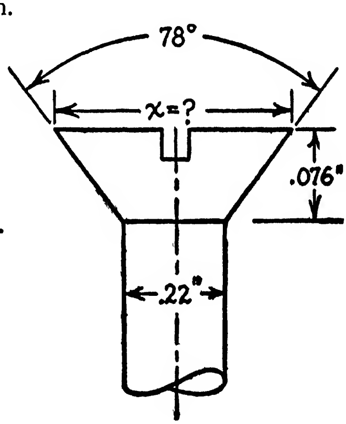

7. If a chord is 18½″ long and the radius of the circle is 14.2″, find the distance of the chord from the center and the central angle.

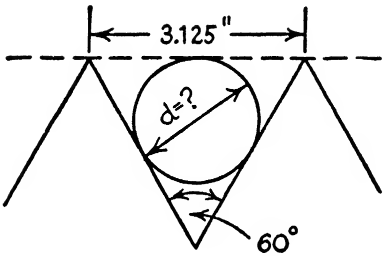
8. Find the perpendicular distance from the center to the sides of a regular hexagon whose sides are each 4.268″ long?

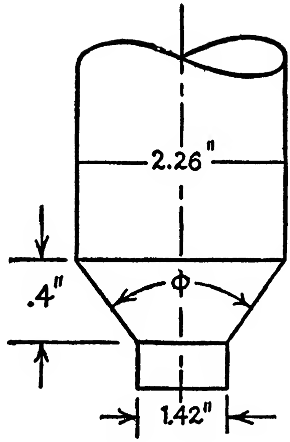



9. Find the radius of a circle inscribed in an equilateral triangle whose perimeter is 45 inches.
10. The side of a regular pentagon is 18.6 inches. Find the radius of the inscribed and circumscribed circles, respectively.
11. Gable rafters 19 ft. long project 1½ ft. beyond the walls of a house and are set with a pitch (angle with the horizontal) of 38°. Find the height of the ridgepole and the width of the house.
12. Find the value of $\phi$ and $x$ from the dimensions given.
13. Find $x$ in the flat-head screw as shown.



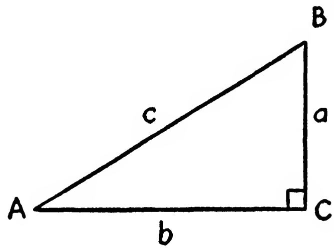



14. Find the diameter $d$ of the wire laid in the groove of the screw thread as shown.
15. Find the angle $\phi$ in the spindle as shown.

## 21. PRACTICAL APPLICATIONS OF RIGHT TRIANGLES

**Solution of Right Triangles.** From the foregoing discussion it should be clear that a right triangle is completely determined if a side and any other part are known. In other words, given the length of a side and any other part, all the other parts can then be found; this is called solving the right triangle. By way of summary, the solution of a right triangle may be effected by the use of one or more of the following fundamental relationships:




(1) $a^2+b^2=c^2$

(2) $\sin A=\dfrac{a}{c}=\cos B$

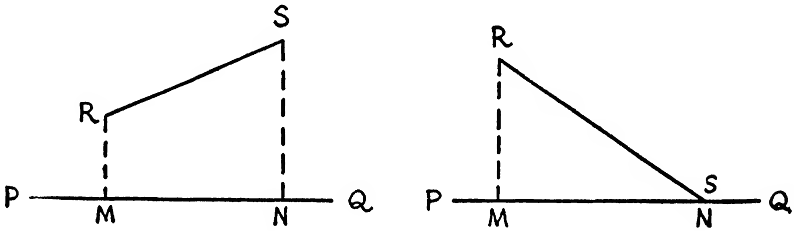
(3) $\cos A=\dfrac{b}{c}=\sin B$

(4) $\tan A=\dfrac{a}{b}=\cot B$

(5) $A+B=90°$

The rest of the present section will deal with practical uses of right triangles. Standard notation, i.e., side "$a$" opposite $\angle A$, side "$b$" opposite $\angle B$, etc., $\angle C$=right angle, will be used consistently throughout.

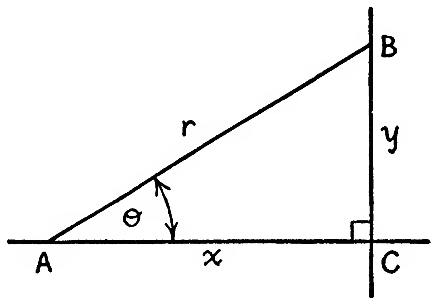
**Projections.** In physics and mechanics frequent use is made of horizontal and vertical projections. As we saw in Chapter III, section 13, the projection of one line segment upon another was the segment on the




second line between the feet of the perpendiculars drawn to the second line from the ends of the first line. Thus, the projection of RS upon PQ in each case here shown is the segment MN. If now, we project an oblique line, such as AB or $r$, upon each of two mutually perpendicular axes, respectively, then the projection of AB upon the horizontal axis is AC, or $x$; and the projection of AB upon the vertical axis is BC, or $y$. These projections, AC and BC, can be expressed as follows:

$$AC = x = r \cdot \cos \theta,$$

$$\text{and } BC = y = r \cdot \sin \theta.$$

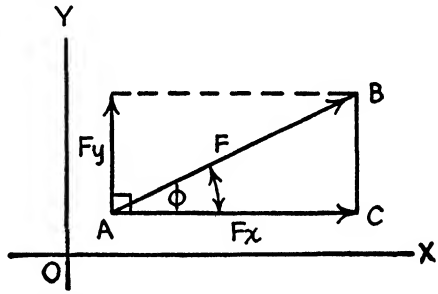



This leads to two simple but important rules concerning projections:

*Rule 1: The horizontal projection of any line segment equals the length of the segment multiplied by the cosine of the angle of inclination, or the angle with the horizontal.*

*Rule 2: The vertical projection of any line segment equals the length of the segment multiplied by the sine of the angle of inclination.*

**Component Forces and Velocities.** These principles may be applied to the component parts of forces or velocities acting obliquely to the horizontal and vertical directions. Thus if AB represents a force $F$ acting at an angle $\phi$ to the horizontal, the two perpendicular component forces $F_x$ and $F_y$ are equivalent, when considered jointly, to the single original force $F$.

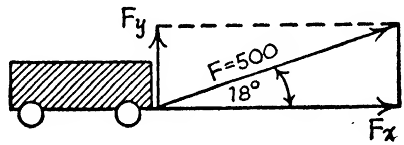



Furthermore, from the preceding paragraph, we now see that

$$F_x = F \cos \phi,$$
$$\text{and } F_y = F \sin \phi.$$

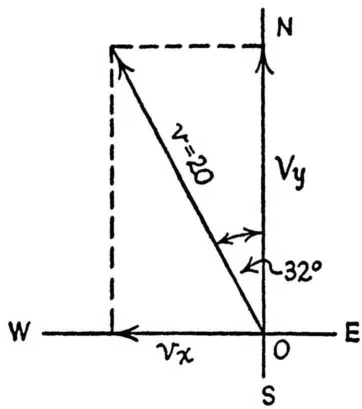
It is also clear that $F_x^2 + F_y^2 = F^2$. From these relations, many simple problems concerning forces and velocities can be solved, as will be shown below.

EXAMPLE 1: A pull of 500 lb. is applied to a cart at an angle of 18° to the horizontal. What is the effective horizontal pull? How much is the force that tends to lift the cart vertically upward?



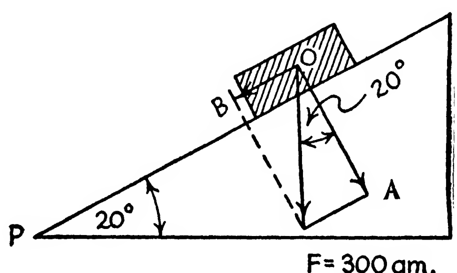

SOLUTION: $F_x = 500(\cos 18°) = 500(.9511) = 476.6$ lb., *Ans.*
$F_y = 500(\sin 18°) = 500(.3090) = 154.5$ lb., *Ans.*

EXAMPLE 2: A ship steering a course 32° west of north is moving at 20 miles an hour. Where will it be one hour after it leaves point O?




SOLUTION: $v_x = 20(\cos 58°) = 20(.5299) = 10.6$ mi. west of point O, *Ans.*
$v_y = 20(\sin 58°) = 20(.8480) = 17.0$ mi. north of point O, *Ans.*

EXAMPLE 3: A block rests upon an incline of 20°. If the block weighs 300 gm., find (a) the pressure exerted perpendicularly upon the inclined plane; and (b) the pull parallel to the plane, tending to cause it to slide down the plane.



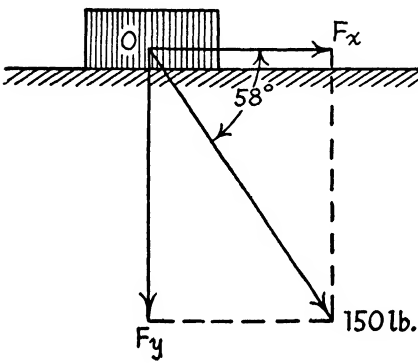
SOLUTION: If the angle at $P=20°$, then $\angle AOF$ also equals $20°$. Hence,

$OA=300(\cos 20°)=300(.9397)=281.9$ gm., *Ans.*

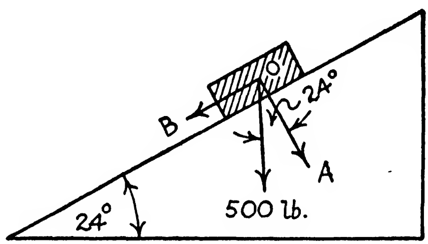
$OB=300(\sin 20°)=300(.3420)=102.6$ gm., *Ans.*

*Exercise 72.*

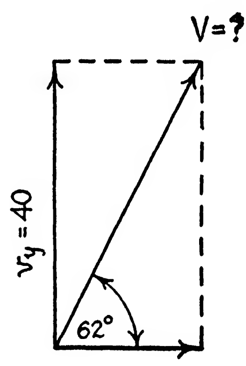
1. An object is moving with a velocity of 200 ft. per minute along a line making an angle of 33° with the horizontal. Find the horizontal component of this velocity.
2. An airplane is flying northeast at the rate of 300 miles per hour. At what rate is it moving eastward? at what rate northward?




3. A force of 150 lb. is applied to a block resting on the horizontal. If the force makes an angle of 58° with the horizontal, what force tends to draw the block horizontally? What force tends to pull it vertically downward?
4. A force of 600 lb. acting in a direction inclined 50° from the vertical is applied to a heavy object. Find (a) the force which tends to move the block horizontally, and (b) the force which tends to lift it vertically.

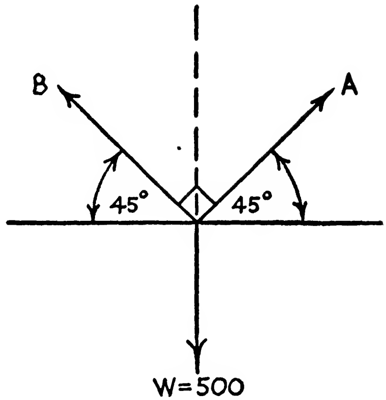



5. A block resting on an incline of 24° weighs 500 lb. Find (a) the perpendicular pressure (OA) exerted by the block against the inclined plane, and (b) the pull (OB) along the plane.
6. Find the velocity of a body moving at an angle of 62° with the horizontal, if the vertical component of its velocity is 40 ft. per second. How fast is it moving horizontally?

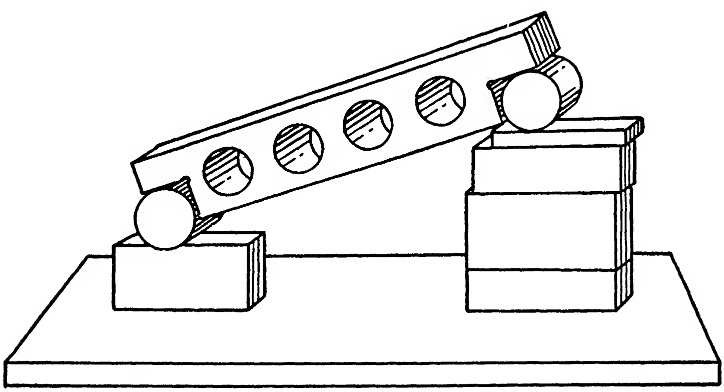


7. The horizontal and vertical components of a force acting on an object are 800 lb. and 600 lb., respectively. Find the original force and its direction of action.

8. A weight $W$ of 500 lb. is suspended from point O by two stout chains, OA and OB. If each chain bears half the share of the total weight $W$, find the upward vertical pull exerted by each chain.




**The Sine Bar.** The sine bar is a device commonly used to facilitate precision angle-measurements. It consists of a very accurately made, heat-treated alloy-steel straight edge, to which are attached two hardened cylinders. All of the surfaces of this straight edge are parallel to each other and to the center line between the two rolls or plugs. These rolls are extremely accurately spaced, and their cylindrical surfaces made square to the measuring surfaces of the bar. The distance between studs on the 5-inch bar is controlled to within $\pm .0002$ inch, and between $\pm .00025$



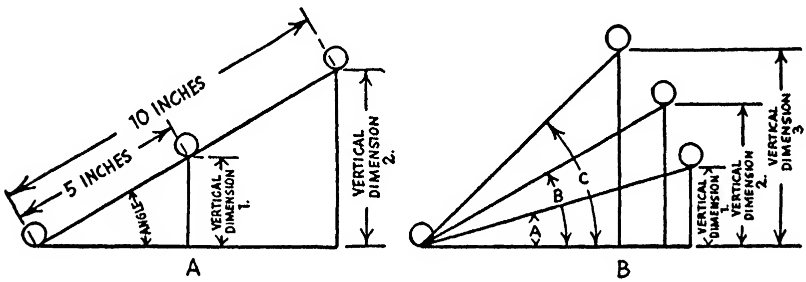
Use of Sine Bar in conjunction with Gage Blocks to obtain precision angular measurement.

inch on the 10-inch bar. These are the two standard sizes of the sine bar. The accompanying illustration shows how the sine bar is used in conjunction with stacks of gage blocks. A 1-inch block is generally used under the lower roll; the stack of blocks under the higher end is adjusted to create the desired angle.

**Theory of the Sine Bar.** The sine bar makes application of the known relation between the sides of a right triangle and its angles.

The right triangle does not exist physically as a solid triangle, but is partly imaginary. The bar itself is the only part of the triangle that actually exists. The base of the triangle is an imaginary line on the surface plate if the lower button is resting on the plate. The vertical leg of the triangle is an imaginary line from the bottom of the sine bar button to the surface plate. It may take the form of a stack of precision gage blocks if the bar is supported by the blocks, but if the bar is clamped to an angle plate, it will probably be just an imaginary line.

The fundamental relation that holds for all angles in a right triangle is that the ratio of the side opposite the angle to the length of the longest side varies with the size of the triangle.

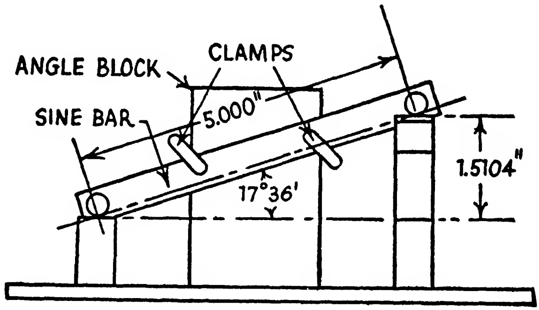



PRINCIPLE OF THE SINE BAR

Now, if the side opposite the right angle is kept constant at some figure, such as 5 inches, then the angle varies directly as the length of the side opposite, and a table of angles and corresponding lengths of opposite sides can be developed. With the aid of such a table, it is necessary to know only the length of the opposite side to determine the angle, and vice versa.

The tables of natural sines found in almost any handbook are developed on the basis of a 1-inch side opposite the right angle, or a 1-inch sine bar. For example, the table of natural sines gives a value of .56184 for the sine of 34° 11′. This means that if the side opposite the right angle is one inch, then the side opposite the angle is .56184 of an inch; and from what has been developed above, it also means that if the side opposite the right angle is 5 inches, then the side opposite the angle is 5 times .56184, or 2.8092 inches. To set up a 5-inch sine bar to an angle of 34° 11′, a stack of gage blocks 2.8092 inches high would be required under one end, with the other resting on the surface plate.

**Setting Up an Angle on the Sine Bar.** Let it be assumed that a part master has an angle of 17° 36′ to be checked with a sine bar. The sine bar is set up to 17° 36′ by noting the value in any sine bar table and building up a stack of blocks so that the opposite side has this value. If the lower end of the bar rests on a block one inch high, then one inch also must be added to the stack supporting the upper end, because it is the difference between the upper and the lower end which determines the opposite side of the triangle.



SETTING UP A GIVEN ANGLE

Any sine bar table would show that a difference of 1.5104 is necessary to produce an angle of 17° 36′. This means that the high end must be set at 2.5104 off the surface plate. With the bar resting on the blocks, it is clamped to the angle iron.

The work is then clamped to the bar so that the 17° 36′ angle on the piece is level with the surface of the plate. The surface is then indicated with a height gage and sensitive lathe indicator to verify the correctness of the angle.

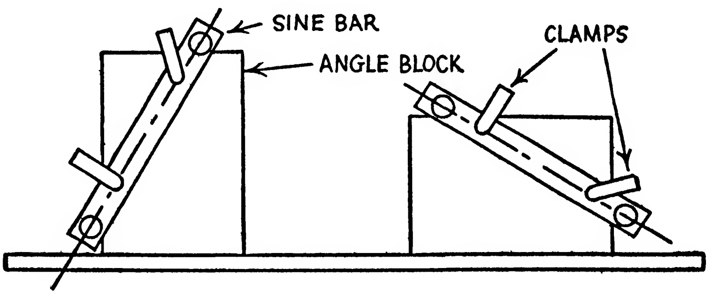
**Setting Up to Measure an Unknown Angle.** This procedure is similar to setting up for checking a known angle. The angle is measured roughly with a bevel protractor, and the stacks of blocks necessary to produce this angle are made up from the figure read in the table.

The work is clamped to the sine bar after it is set up, and indicated with a height gage and indicator. The error noted is corrected by increasing or decreasing the height of the stack of blocks under the high end. When the indicator shows the surface to be level, the length of the stack is noted. From this, the stack at the lower end is subtracted, and the net value is the opposite side. Referring to a sine bar table, the value nearest this dimension represents the angle measured to the nearest minute. Although it might be possible to compute the number of seconds, for all practical purposes "to the nearest minute" is considered satisfactory.

**Applications of the Sine Bar.** The application of the sine bar generally is reserved for measurements which require that an angle be determined closer than five minutes, which is the limit of accuracy of the bevel pro-

tractor. Of course, with the sine bar it is a simple matter to determine an angle within one minute.

The sine bar is most accurate when measuring small angles, since a small change in the angle at or near the horizontal produces a greater change in the vertical dimension measured by the blocks. As the angle approaches 90 degrees, it is easy to see that the change in height corresponding to a given change in angle is much smaller. For this reason a sine bar should not be set up to measure an angle over 60 degrees, but the setup should be changed so that the complement of the angle can be measured. The complement of the angle is the difference between the angle and 90 degrees, and is obtained by tipping over the angle block to which the sine bar is set up.




MEASURING THE COMPLEMENT

The reason for tipping the bar over and measuring the complement of a large angle instead of the angle directly is illustrated by the following example. An angle of 80 degrees has a sine of .98481, and the opposite side, using a 5-inch bar, would be 4.92405. An angle of 80 degrees, 1 minute would have an opposite side of 4.92430. This means that one minute is represented by a difference of .00025 of an inch in the height of the blocks.

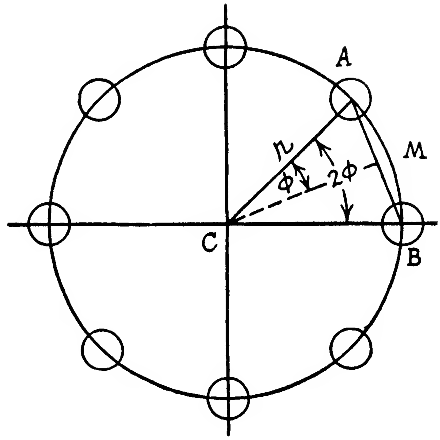
If this angle is turned over on its side it becomes an angle of 10 degrees. A table of sine bar values quickly shows .86825 as the height of the stack of blocks necessary to produce an angle of 10 degrees. An angle of 10 degrees, 1 minute is produced by a stack .86965 of an inch high. One minute is represented by a difference of .0014, which means that at 10 degrees, eliminating the error in the tools themselves, a measurement is more than five times as accurate as one made at 80 degrees. This is the reason why sine bar tables usually stop at 60 degrees, and the inspector or toolmaker using a sine bar will always try to set up an angle over 60 degrees so that he checks the complement instead of the angle itself.

**Spacing Holes on a Circle.** In order to find the distance from center to center between holes spaced on a circle, it is only necessary to find the

length of the chord (AB) joining the centers A and B. If the angle of arc AB is $2\phi$, then clearly,

$$\sin \phi = \frac{AM}{r},$$

or $AM = r \sin \phi$; since $AB = 2(AM)$, then $AB = 2r \sin \phi$; or, in terms of the diameter, the distance between adjacent holes equals $d \sin \phi$. In practice, bolt holes are usually spaced at equal distances from one another; i.e., so that their arcs have equal central angles. The circle drawn through the centers of the holes is called the *bolt circle,* and its diameter is, of course, less than that of the rim or outer edge of the disc, wheel, etc.



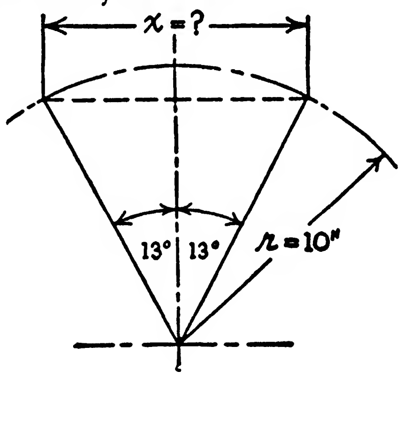

Example: Find the distance between 12 equally spaced holes on a 20″ circle.

Solution: $360° \div 12 = 30°$,
$\frac{1}{2} \times 30° = 15°$.

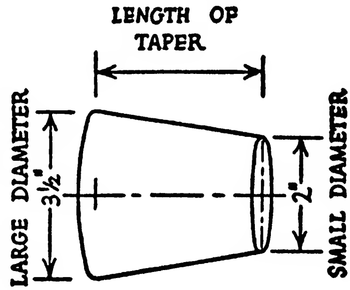
Hence distance between centers $= d \sin 15°$
$= 20(.2588) = 5.176''$, *Ans.*

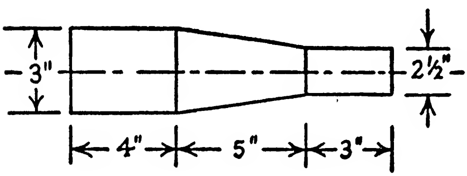
*Exercise 73*

1. If 8 equally spaced holes are to be drilled on a 14-in. circle, find the distance between the centers of any two adjacent holes.
2. Three holes are to be drilled 120° apart on a 12-in. bolt circle. What is the center to center distance between any two holes?
3. Find the center to center distance $x$ between the two holes shown, if they are spaced 26° apart.
4. A bronze casting 3′ 8″ in diameter is to have 6 holes drilled in it. If the bolt circle on which the holes are to be placed is 1½″ away from the outer edge of the casting, how far apart must the holes be placed?



**Tapers.** In designing a conical piece of work such as here shown, the amount of the slope of the sides is called the *taper*. Such a tapered piece is actually the frustum of a cone. The amount of taper is defined as the difference in diameter per unit length of the tapered part. Frequently only part of an entire piece is tapered, the rest of it being cylindrical. If $D$=large diameter, $d$=small diameter, $L$=length of tapered part only, T. T.=total taper, and T= the taper per inch, then, in the second diagram:

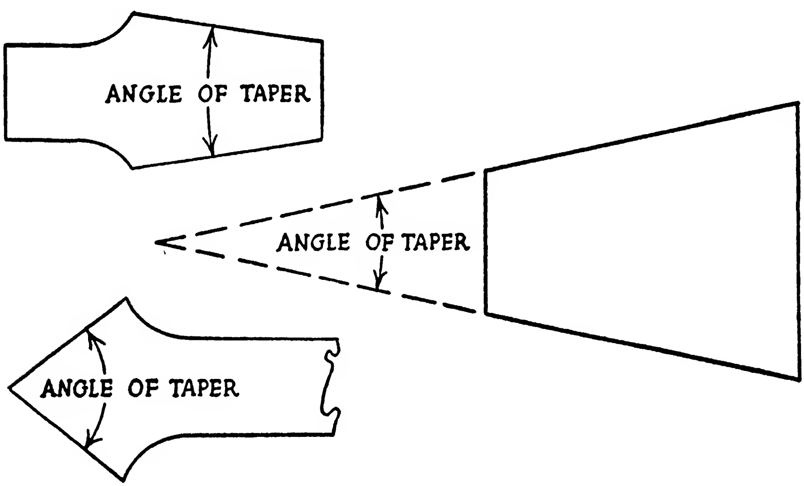


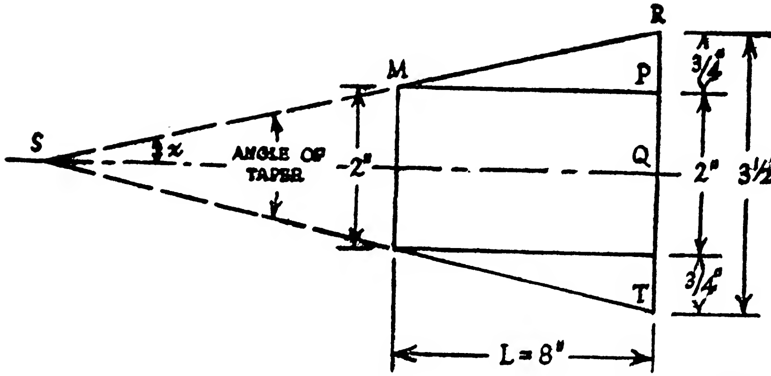

T. T.$=D-d=3''-2\tfrac{1}{2}''=\tfrac{1}{2}''$;

and $T=\dfrac{D-d}{L}=\dfrac{\tfrac{1}{2}}{5}=0.1''$ per inch.

**Angle of Taper.** The taper on a piece of work may be expressed as so many inches per inch, or as many inches per foot. It may also be described in terms of the angle included between the sides (or the prolongation of the sides) of the piece of work.




Note carefully that the angle between a sloping side (or its extension) and the center line represents only *half* the taper angle. Computation of tapers and taper angles involves the tangent of an angle, very simply.



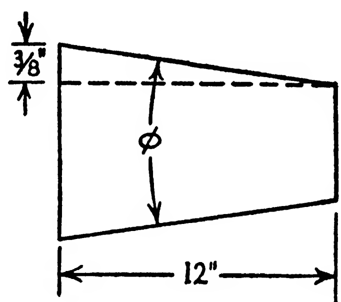

It is clearly seen that $\triangle$ RMP is similar to $\triangle$ RSQ, and that $\angle$RMP$=$ $\angle$RSQ$=$½ (taper angle). Hence:

$$\tan x=\frac{RP}{MP}=\frac{\frac{1}{2}(D-d)}{L}=\frac{\frac{1}{2}(3\frac{1}{2}-2)}{8}=\frac{\frac{3}{4}}{8}=.0938, \text{ or } x=5°21\tfrac{1}{2}'.$$

taper angle$=2\times5°21\frac{1}{2}'=10°43'$.

**Example 1:** What is the angle of taper in a piece of work having a taper of ¾″ per foot?

**Solution:**




$$\tan(\tfrac{1}{2}\text{ taper angle})=\frac{\frac{1}{2}(\text{taper})}{\text{length in inches}}$$

$$\tan\frac{\phi}{2}=\frac{\frac{1}{2}\times\frac{3}{4}}{12}=\tfrac{1}{2}\times\tfrac{3}{4}\times\tfrac{1}{12}=.0313.$$

$$\frac{\phi}{2}=1°48'$$

$\phi=2\times1°48'=3°36'$, *Ans.*

**Example 2:** What is the taper per foot of a piece of work if the angle of taper is 8°30′?

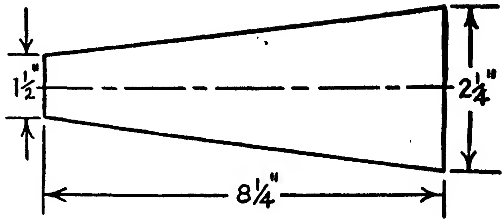
**Solution:** ½(taper angle)$=\frac{1}{2}\times8°30'=4°15'$.
$\tan 4°15'=.0743$.

Hence taper per inch$=2\times.0743=.1486''$,
and taper per foot$=12\times.1486=1.7832''$, *Ans.*

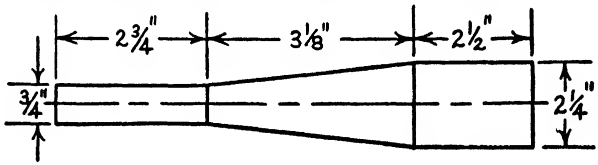
In other words, to find the taper per inch when the taper angle is given, we find the tangent of half the taper angle; this gives the taper from the center line in inches per inch. To find the total taper per inch, we multiply this result by 2. To find the total taper per foot, multiply the total taper per inch by 12.

*Exercise 74.*

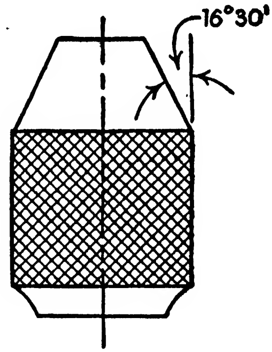
1. Find the angle of taper of the piece shown in the accompanying diagram.



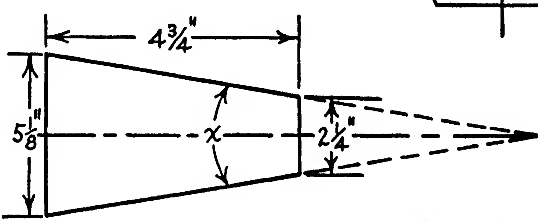

2. Find the taper angle in the following piece of work.




3. In the tapered drill chuck shown, find (a) the angle of taper, and (b) the taper per inch.

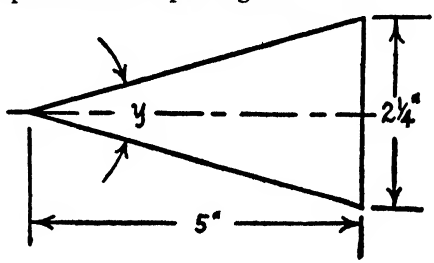



4. Determine angle $x$ in the tapered piece shown.



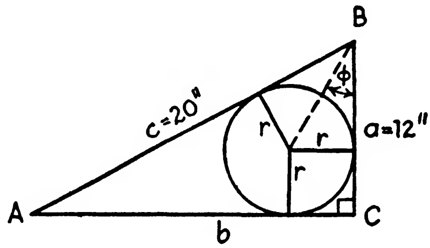

5. What is the taper per foot of a piece whose taper angle is 12°10′?

6. Find $\angle y$; if the length which is now 5″ were doubled, what would $\angle y$ then be equal to?



**Miscellaneous Applications.** Many problems of design in the machine shop are readily solved by making use of trigonometric relationships. Study the following and you will see how helpful trigonometry can be.

EXAMPLE 1: Find the diameter of the circle inscribed in a right triangle, one of whose sides is 12″ and whose hypotenuse is 20″; also, find $\phi$.




SOLUTION: It can be shown that, when a circle is inscribed in any right triangle, the following relation holds:

hypotenuse+diameter=sum of the sides;

$$c+2r=a+b$$

or, diameter $=a+b-c$

Hence, $b=\sqrt{(20)^2-(12)^2}=16''$

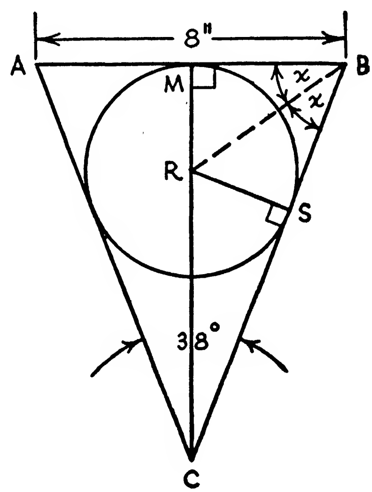
and diameter $=12+16-20=8''$, *Ans.*

$\tan B=16/12=1.3333$

$B=53°8'$

$\phi=\frac{1}{2}(B)=26°34'$, *Ans.*

EXAMPLE 2: Given a circle inscribed in an isosceles triangle ABC. Find the diameter of the circle if AB =8″ and ∠ABC=38°.




SOLUTION:

$\angle MCB=19°$; $MB=4''$;

$\angle x=\frac{1}{2}(90°-19°)=35°30'$

$MR=(MB)(\tan x)$

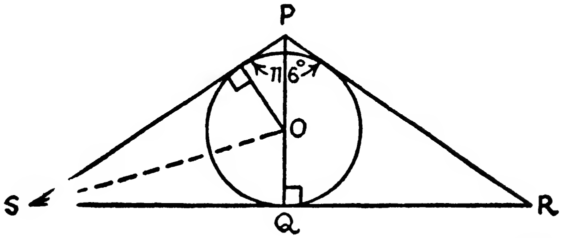
$=(4)(\tan 35°30')$

$=(4)(.7133)=2.8532$

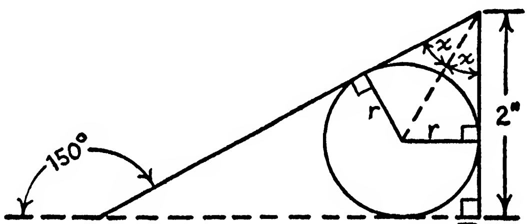
diameter $=2\times 2.8532=5.706''$, *Ans.*

*Exercise 75.*

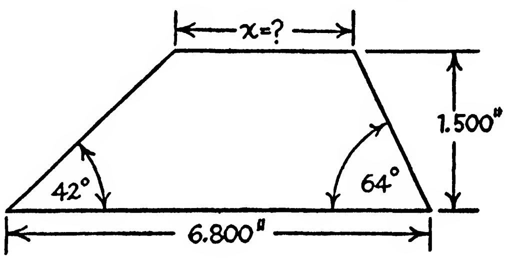
1. Find the diameter of a circle inscribed in a right triangle whose hypotenuse is 18″ and whose longer side is 10″.
2. What is the diameter of a circle inscribed in a right triangle whose shorter side is 6″, if the angle adjacent to that side is 42°?
3. If PQ=12″, and ∠SPR=116°, find the radius of the inscribed circle.

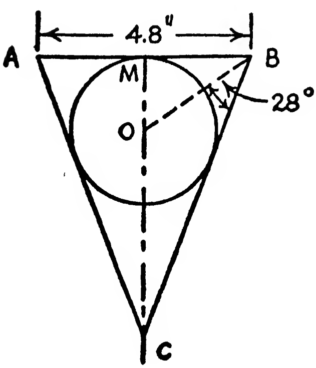



4. Find the diameter of the inscribed circle in the diagram shown.



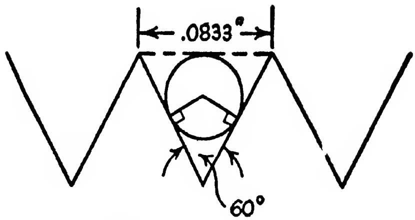

5. Find the shorter base of the trapezoid in the given diagram.



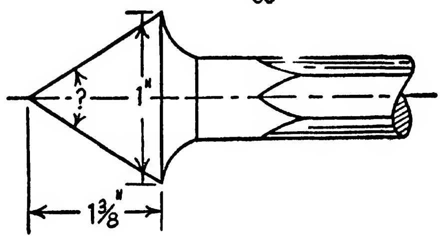




6. If ∠OBC=28°, find (a) the diameter of the inscribed circle, and (b) the depth MC.

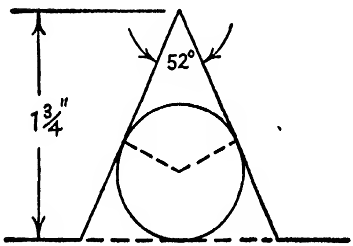
7. Find the diameter of the wire flush with the top of the 60°-screw thread shown, if the distance between two peaks is .0833″.

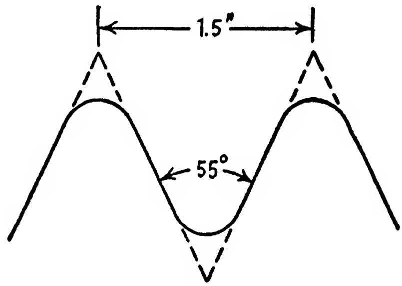



8. Find the included angle in the point of the countersink tool shown.

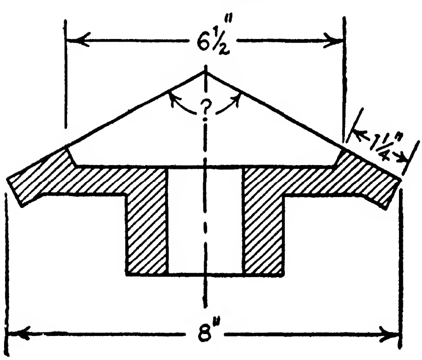


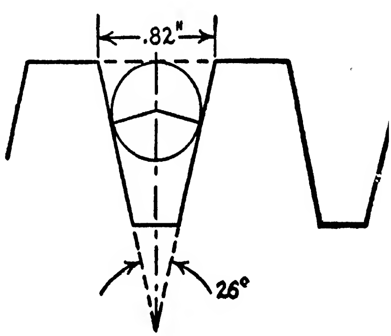

9. Find the diameter of the plug required to fit the 52° angle shown.



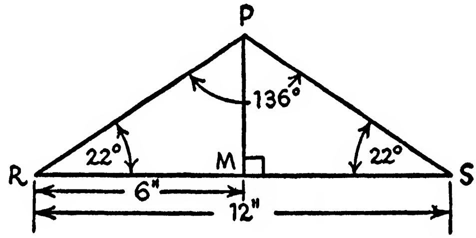

10. The special 55°-thread screw has a depth equal to ⅔ of the depth of the triangle formed when the sides of the thread are extended. Find the depth of the thread when the distance from top to top is 1.5″.
11. Find the included angle of the bevel gear blank shown below.
12. Find the diameter of the wire inserted in the worm thread with the 26°-included angle as shown.



## 22. SOLUTION OF OBLIQUE TRIANGLES

**Oblique Triangles.** By an oblique triangle is meant a triangle none of whose angles is a right angle. To solve an oblique triangle means to find the remaining sides and angles when some of these parts are known. This can sometimes be done by breaking the triangle down into right triangles; for example, if we wish to find the sides of an isosceles triangle whose base is 12″ and whose vertex angle in 136°, we draw PM perpendicular to RS, which divides the given triangle into two right triangles from which we can readily find PR. Thus:




$$\frac{RM}{RP}=\sin 68°$$

$$RP=\frac{RM}{\sin 68°}=\frac{6}{.9272}=6.5, \textit{ Ans.}$$

However, it is not always convenient or possible to do this with oblique triangles, and so other methods must be used. We shall continue to use the standard notation for the sides and angles of a triangle, exactly as was done in the case of right triangles.

**Functions of an Obtuse Angle.** Before introducing these new methods, however, it is necessary to show how to find the functions of an obtuse angle. We already know how to find the functions of $(90°-a)$; thus

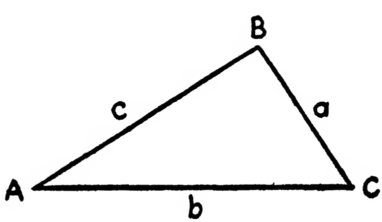
$$\sin (90°-a)=\cos a$$
$$\cos (90°-a)=\sin a$$
$$\tan (90°-a)=\cot a$$

It can be proved, although we shall not stop to do it here, that the transformations for finding functions of $(90°+a)$ and $(180°-a)$ are as follows:

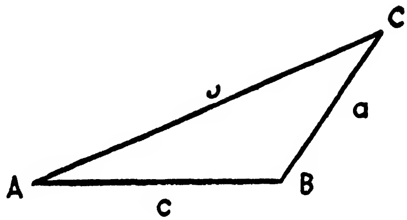
$$\sin (90°+a)= \cos a \qquad \sin (180°-a)= \sin a$$
$$\cos (90°+a)=-\sin a \qquad \cos (180°-a)=-\cos a$$
$$\tan (90°+a)=-\cot a \qquad \tan (180°-a)=-\tan a$$

**Law of Sines.** A convenient trigonometrical relationship which may be utilized in solving oblique triangles is the *law of sines,* which may be stated as follows:

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$



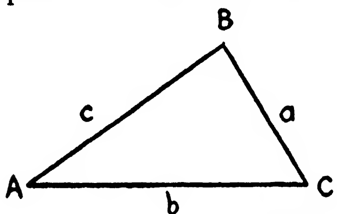
Expressed in words: *any side of a triangle is to the sine of the angle opposite that side as any other side is to the sine of its opposite angle.* Or, it may be written in another form, viz.:

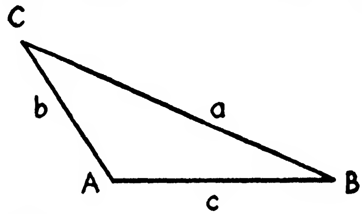



$$\frac{a}{b}=\frac{\sin A}{\sin B};\quad \frac{a}{c}=\frac{\sin A}{\sin C};\quad \frac{b}{c}=\frac{\sin B}{\sin C}.$$

That is, the ratio of any two sides of a triangle is equal to the ratio of the sines of the angles opposite them, respectively.

**Law of Cosines.** This law states that in any triangle, the square of any side equals the sum of the squares of the other two sides diminished by the product of those two sides and the cosine of their included angle.




$$a^2=b^2+c^2-2bc\cos A$$
$$b^2=a^2+c^2-2ac\cos B$$
$$c^2=a^2+b^2-2ab\cos C.$$

Or, solving for the angles, we get:

$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

$$\cos B=\frac{a^2+c^2-b^2}{2ac}$$

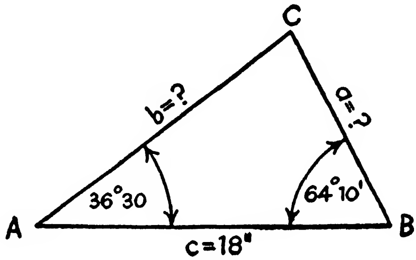
$$\cos C=\frac{a^2+b^2-c^2}{2ab}.$$

**Types of Problems in Solving Oblique Triangles.** It is convenient to consider four types of problems, or sets of given conditions, when dealing with oblique triangles, viz.:

I. Given one side and any two angles.
II. Given two sides and the included angle.
III. Given two sides and an angle opposite one of them.
IV. Given the three sides only.

We shall explain the method used to solve each of these four types; in the illustrative problems we shall use the tables in the back of the book.

**TYPE I: Given One Side and Any Two Angles.** In this case the two given angles may be adjacent to the given side (a.s.a.) or they may not be (s.a.a.); it makes no difference, however, since if any two angles of a triangle are known, the third may be found immediately by subtracting their sum from 180°.




EXAMPLE 1: Given: $C=18''$, $A=36°30'$, and $B=64°10'$. Find the remaining sides.

SOLUTION: $C=180°-(36°30'+64°10')=79°20'$

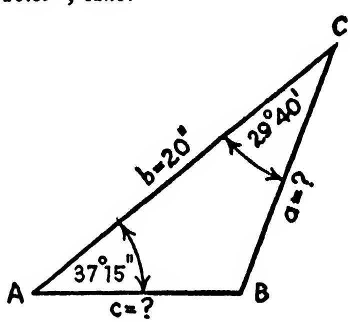
$$\frac{b}{c}=\frac{\sin B}{\sin C} \qquad \frac{a}{c}=\frac{\sin A}{\sin C}$$

$$\frac{b}{18}=\frac{\sin 64°10'}{\sin 79°20'} \qquad \frac{a}{18}=\frac{\sin 36°30'}{\sin 79°20'}$$

$$\frac{b}{18}=\frac{.9001}{.9827} \qquad \frac{a}{18}=\frac{.5948}{.9827}$$

$$b=\frac{(18)(.9001)}{.9827} \qquad a=\frac{(18)(.5948)}{.9827}$$

$b=16.49''$, *Ans.* $\qquad a=10.89''$, *Ans.*




EXAMPLE 2: Given: $b=20''$, $A=37°15'$, and $C=29°40'$. Solve the triangle.

SOLUTION: $B=180°-(37°15'+29°40')=113°5'$

$\sin B=\sin 113°5'=\sin(90°+23°5')=\cos 23°5'$

$$\frac{c}{b}=\frac{\sin C}{\sin B} \qquad \frac{a}{b}=\frac{\sin A}{\sin B}$$

$$\frac{c}{20}=\frac{\sin 29°40'}{\cos 23°5'} \qquad \frac{a}{20}=\frac{\sin 37°15'}{\cos 23°5'}$$

$$\frac{c}{20}=\frac{.4950}{.9199}$$

$$c=\frac{(20)(.4950)}{.9199}$$

$c=10.76''$, *Ans.*

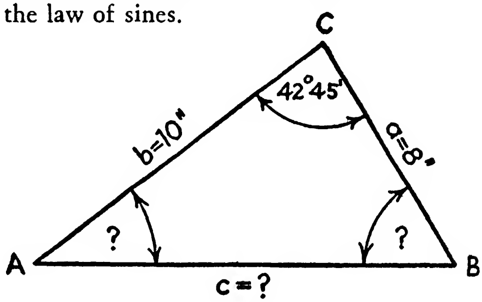
$$\frac{a}{20}=\frac{.6053}{.9199}$$

$$a=\frac{(20)(.6053)}{.9199}$$

$a=13.16''$, *Ans.*

**TYPE II: Given Two Sides and the Included Angle.** In this case we use the law of cosines, as well as the law of sines.

EXAMPLE 1: Given: $a=8''$, $b=10''$, and $C=42°45'$. Solve the triangle.




SOLUTION: $c^2=a^2+b^2-2ab\cos C$

$c^2=8^2+10^2-2(8)(10)(\cos 42°45')$

$c^2=64+100-(160)(.7343)=46.512$

$c=\sqrt{45.87}=6.82''$, *Ans.*

Using the sine law to find $A$, we have:

$$\frac{a}{\sin A}=\frac{c}{\sin C}$$

$$\sin A=\frac{a\sin C}{c}=\frac{(8)(.6788)}{6.82}=.7962$$

$A=52°46'$, *Ans.*

Angle $B$ may be found by difference:

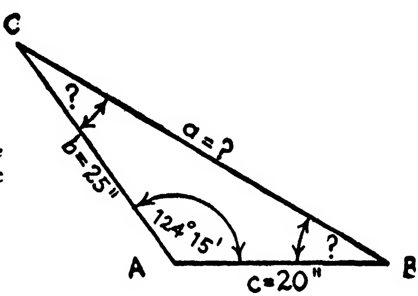
$B=180°-(42°45'+52°46')=84°29'$, *Ans.*

As a check, use the law of sines:

$$\sin B=\frac{b\sin C}{c}=\frac{(10)(.6788)}{6.82}=.9953$$

from table, $\sin 84°29'=.9954$, Check.

EXAMPLE 2: Given: $b=25''$, $c=20''$, and $A=124°15'$. Solve the triangle.



SOLUTION: $a^2=b^2+c^2-2bc \cos A$

$a^2=(25)^2+(20)^2-2(25)(20)(\cos 124°15')$.

But, $\cos 124°15'=\cos (90°+34°15')=-\sin 34°15'$

Hence, $a^2=625+400-(1000)(\sin 34°15')$

$a^2=625+400-(1000)(.5628)=1587.8$

$a=\sqrt{1587.8}=39.847''$, *Ans.*

$$\frac{c}{\sin C}=\frac{a}{\sin A}$$

$$\sin C=\frac{c \sin A}{a}=\frac{(20)(\sin 124°15')}{39.847}.$$

But, $\sin 124°15'=\sin (90°+34°15')=\cos 34°15'=.8266$

$$\sin C=\frac{(20)(.8266)}{39.847}=.4149$$

$C=24°31'$, *Ans.*

$B=180°-(124°15'+24°31')=31°12'$, *Ans.*

Check: $\sin B=\dfrac{b \sin A}{a}=\dfrac{(25)(.8266)}{39.847}=.5186$

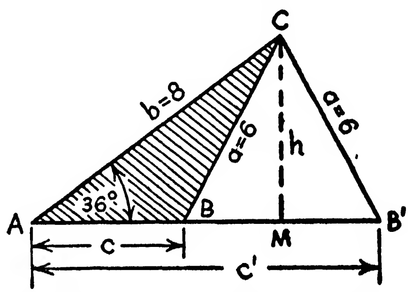
from the table, $\sin 31°12'=.5180$, Check.

*Exercise 76.*

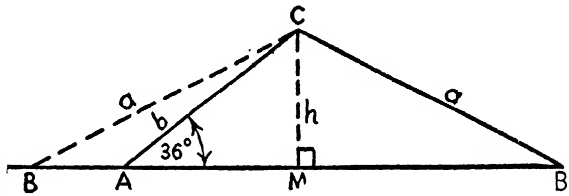
Solve the following oblique triangle, given the parts specified:

1. $A=42°30'$, $B=54°40'$, $c=20''$
2. $A=39°10'$, $C=112°$, $b=24''$
3. $A=38°45'$, $B=62°12'$, $a=18.2''$
4. $a=16''$, $c=10''$, $B=47°20'$
5. $a=20''$, $b=12''$, $C=135°$

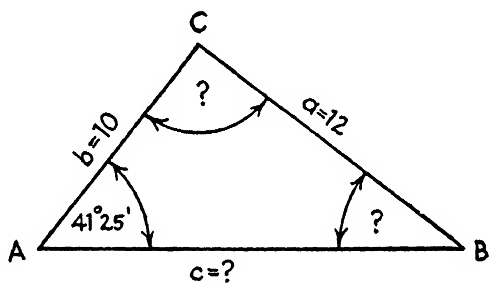
**TYPE III: Given Two Sides and an Angle Opposite One of Them.** This case is a bit different from the others in that there is the possibility of two solutions, one solution, or no solution, depending upon the conditions of the problem. Consider the following diagram; given $a=6$, $b=8$, and $A=36°$. By studying the diagram it will be realized that not only the shaded triangle ABC, but also the large triangle AB′C (where CB=CB′) will satisfy the given conditions. In other words, if $a>h$ (or $b \sin A$) but $<b$, then *two* solutions are possible, viz.,



△ABC and △AB′C. If $a=h$ (or $b \sin A$), then only *one* solution is possible, viz., right triangle AMC. If $a<h$, *no* solution is possible, for no triangle exists in that event. Finally, if $a>h$ and also $a>b$, then again only *one* solution is possible, viz., △ACB′, since △ACB, while it contains the given parts $a$ and $b$, does *not* contain $\angle A$ (36°) as one of its interior angles, and hence does not represent a required solution. This will now be further illustrated.




EXAMPLE 1: Given: $a=12$, $b=10$, and $A=41°\ 25'$. Solve the triangle.




SOLUTION: By constructing the triangle approximately, it is obvious that only one solution is possible.

$$\sin B=\frac{b \sin A}{a}=\frac{(10)(\sin 41°25')}{12}$$

$$\sin B=\frac{(10)(.6615)}{12}=.5513$$

$B=33°27'$, *Ans.*

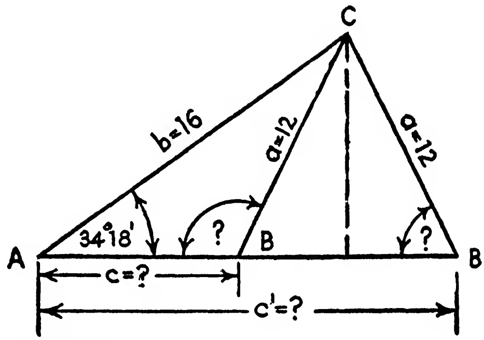
$C=180°-(41°25'+33°27')=105°8'$, *Ans.*

$$c=\frac{a \sin C}{\sin A}=\frac{(12)(\sin 105°8')}{\sin 41°25'}=\frac{(12)(\sin 74°52')}{\sin 41°25'}$$

$$c=\frac{(12)(.9653)}{.6615}=17.511, \textit{ Ans.}$$

**Example** 2: Given: $a=12$, $b=16$, and $A=34°18'$. Solve the triangle.

**Solution**: By constructing the triangle approximately, it is seen that two triangles are possible—either △ABC or △AB′C. We shall solve △AB′C first.




$$\sin B'=\frac{b \sin A}{a}=\frac{(16)(.5635)}{12}=.7180$$

$B'=45°54'$, *Ans.*

$C=180°-(A+B)=99°48'$, *Ans.*

$$c'=\frac{a \sin C}{\sin A}=\frac{(12)(\sin 99°48')}{\sin 34°18'}=\frac{(12)(.9854)}{.5635}$$

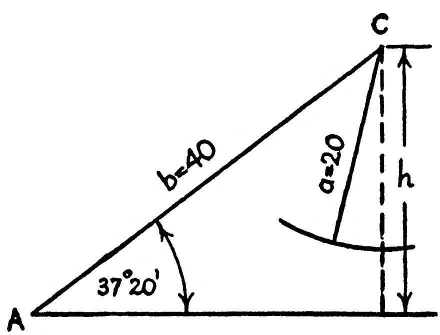
$c'=20.984$, *Ans.*

Now, to find $c$ in the *second* solution, we note that ∠CBA=180° —45°54′=134°6′, since △ CBB′ is isosceles and ∠CBB′=∠CB′B= 45°54′. Also note that ∠ABC therefore equals 11°36′.

Thus, $c=\frac{b \sin C}{\sin B}=\frac{(16)(\sin 11°36')}{\sin 134°6'}$; $c=\frac{(16)(.2011)}{.7181}=4.481$, *Ans.*

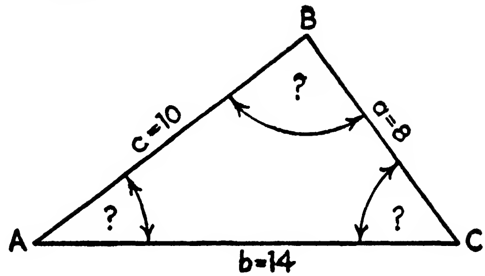
**Example** 3: Given: $a=20$, $b=40$, and $A=37°20'$. Solve the triangle.

**Solution**: In attempting to construct the triangle approximately, it will be seen that, since $h=b\ (\sin 37°20')=(40)(.6065)=24.26$, the length of $a$ is $<h$ and so the triangle cannot be constructed with the parts given; thus no solution is possible.




**TYPE IV: Given the Three Sides Only.** When only the three sides and no angles are given, the law of cosines is used once more. In this case a triangle is always possible so long as the sum of any two of the given sides exceeds the third.

EXAMPLE: Given: $a=8$, $b=14$, $c=10$. Solve the triangle.




SOLUTION:

$$\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{(14)^2+(10)^2-(8)^2}{2(14)\ (10)}=\frac{232}{280}=.8286$$

$A=34°3'$, *Ans.*

$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{(8)^2+(14)^2-(10)^2}{2(8)(14)}=\frac{160}{224}=.7143$$

$C=44°25'$, *Ans.*

$B=180°-(A+C)=101°32'$, *Ans.*

Check:

$$\sin B=\frac{b \sin A}{a}=\frac{(14)(.5599)}{8}=.9798$$

$\sin 101°32'=\cos 11°32'=.9798$

*Exercise 77.*

Solve the triangles, given the following parts:

1. $A=66°$, $a=28''$, $b=16''$
2. $a=10''$, $b=12''$, $c=18''$
3. $b=5.6''$, $c=2.4''$, $B=110°$
4. $a=16.5''$, $b=12.8''$, $c=20''$
5. $A=27°30'$, $a=10''$, $c=18''$

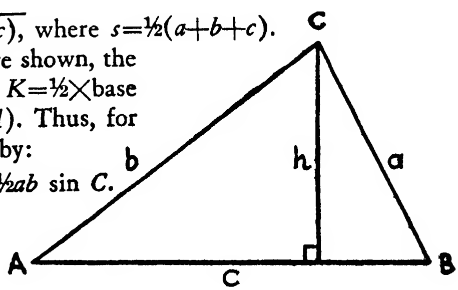
**Area of Triangles.** We have already seen, in the previous chapter, that the area of a triangle ($K$) is given by the following expressions:

(1) $K=\frac{1}{2}$ (base) (altitude);

(2) $K=\sqrt{s(s-a)(s-b)(s-c)}$, where $s=\frac{1}{2}(a+b+c)$.

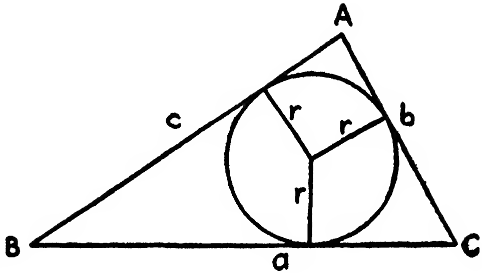
By considering the diagram here shown, the altitude $h=b \sin A$; hence area $K=\frac{1}{2}\times$base $\times$ altitude, or $K=\frac{1}{2}c(b \sin A)$. Thus, for any triangle, the area is given by:

$K=\frac{1}{2}bc \sin A=\frac{1}{2}ac \sin B=\frac{1}{2}ab \sin C.$



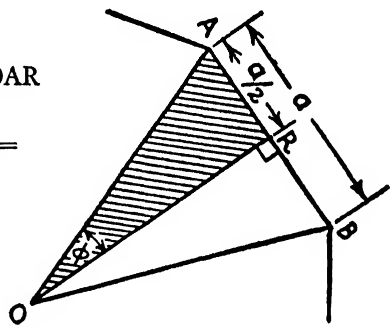
From (2) above, without actually proving it here, it can also be shown that the area of a triangle in terms of its sides and the radius of the inscribed circle is given by:

$K = s \times r$, where $r$=radius of the inscribed circle, and $s = \frac{1}{2}(a+b+c)$.




**Area of Regular Polygons.** If the side of a regular polygon is $a$, and there are $n$ such sides, then $\phi = \frac{360°}{2n} = \frac{180°}{n}$, and OR=(AR) (cot $\phi$) $= \frac{a}{2}$ (cot $\phi$). Hence the area of $\triangle$OAR $= \frac{1}{2}(AR)(OR) = \frac{1}{2}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)(\cot \phi) = \frac{a^2}{8} \cot \phi$. But as there are $2n$ such triangles in the entire regular polygon, the area of the polygon thus equals $2n \times \frac{a^2}{8} \cot \phi$, or

$$K = \frac{1}{4} n a^2 \cot \frac{180°}{n}.$$




In a similar way it can be shown that the following relations also hold for regular polygons having $n$ sides:

(1) Area of a regular polygon circumscribed about a circle whose radius is $R$:

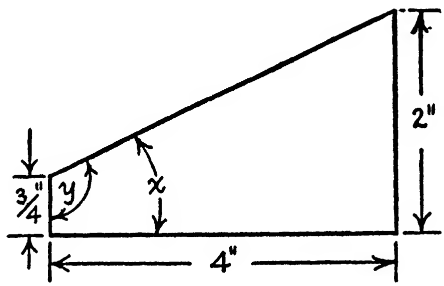
$$K = nR^2 \tan \frac{180°}{n}.$$

(2) Area of a regular polygon inscribed in a circle whose radius is $r$:

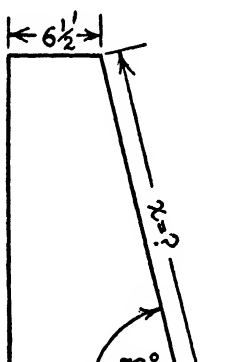
$$K = \frac{1}{2} n r^2 \sin \frac{360°}{n}.$$

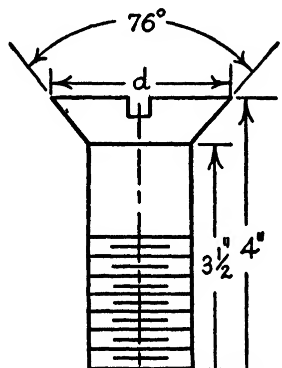
*Exercise 78.*

1. Find the area of an oblique triangle in which $A=34°30'$, $c=18.2''$, and $b=22.5''$.
2. Find the area of a parallelogram whose sides are 20″ and 32″, and one of whose angles is 42°20′.
3. Find the area of a regular polygon of 10 sides inscribed in a circle whose diameter is 40 cm.
4. Find the radius of a circle inscribed in a triangle whose sides are 12″, 18″ and 22″.
5. What is the area of a regular polygon of 12 sides, if each side of the polygon is 6″ long?
6. Find the number of degrees and minutes in angles $x$ and $y$ in the accompanying figure.
7. A parallelogram has two sides equal to 6″ and 9″, respectively, and the included angle is 39°. Find the shorter diagonal.




8. What is the length of the sloping side of the cross section of a con crete embankment if it makes an angle of 78° with the base, which is 15 ft. wide, and the top of the embankment is 6½ ft. wide?




9. Find the diameter $d$ of the bolt with dimensions as shown.
10. The sides of a parallelogram are 10″ and 16″ long, respectively. If the longer diagonal is 20″, what are the angles of the parallelogram?

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| ′ | 0° | | 1° | | 2° | | 3° | | 4° | | ′ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | sin | cos | sin | cos | sin | cos | sin | cos | sin | cos | |
| 0 | 0000 | 1.000 | 0175 | 9998 | 0349 | 9994 | 0523 | 9986 | 0698 | 9976 | 60 |
| 1 | 0003 | 1.000 | 0177 | 9998 | 0352 | 9994 | 0526 | 9986 | 0700 | 9975 | 59 |
| 2 | 0006 | 1.000 | 0180 | 9998 | 0355 | 9994 | 0529 | 9986 | 0703 | 9975 | 58 |
| 3 | 0009 | 1.000 | 0183 | 9998 | 0358 | 9994 | 0532 | 9986 | 0706 | 9975 | 57 |
| 4 | 0012 | 1.000 | 0186 | 9998 | 0361 | 9993 | 0535 | 9986 | 0709 | 9975 | 56 |
| 5 | 0015 | 1.000 | 0189 | 9998 | 0364 | 9993 | 0538 | 9986 | 0712 | 9975 | 55 |
| 6 | 0017 | 1.000 | 0192 | 9998 | 0366 | 9993 | 0541 | 9985 | 0715 | 9974 | 54 |
| 7 | 0020 | 1.000 | 0195 | 9998 | 0369 | 9993 | 0544 | 9985 | 0718 | 9974 | 53 |
| 8 | 0023 | 1.000 | 0198 | 9998 | 0372 | 9993 | 0547 | 9985 | 0721 | 9974 | 52 |
| 9 | 0026 | 1.000 | 0201 | 9998 | 0375 | 9993 | 0550 | 9985 | 0724 | 9974 | 51 |
| 10 | 0029 | 1.000 | 0204 | 9998 | 0378 | 9993 | 0552 | 9985 | 0727 | 9974 | 50 |
| 11 | 0032 | 1.000 | 0207 | 9998 | 0381 | 9993 | 0555 | 9985 | 0729 | 9973 | 49 |
| 12 | 0035 | 1.000 | 0209 | 9998 | 0384 | 9993 | 0558 | 9984 | 0732 | 9973 | 48 |
| 13 | 0038 | 1.000 | 0212 | 9998 | 0387 | 9993 | 0561 | 9984 | 0735 | 9973 | 47 |
| 14 | 0041 | 1.000 | 0215 | 9998 | 0390 | 9992 | 0564 | 9984 | 0738 | 9973 | 46 |
| 15 | 0044 | 1.000 | 0218 | 9998 | 0393 | 9992 | 0567 | 9984 | 0741 | 9973 | 45 |
| 16 | 0047 | 1.000 | 0221 | 9998 | 0396 | 9992 | 0570 | 9984 | 0744 | 9972 | 44 |
| 17 | 0049 | 1.000 | 0224 | 9997 | 0398 | 9992 | 0573 | 9984 | 0747 | 9972 | 43 |
| 18 | 0052 | 1.000 | 0227 | 9997 | 0401 | 9992 | 0576 | 9983 | 0750 | 9972 | 42 |
| 19 | 0055 | 1.000 | 0230 | 9997 | 0404 | 9992 | 0579 | 9983 | 0753 | 9972 | 41 |
| 20 | 0058 | 1.000 | 0233 | 9997 | 0407 | 9992 | 0581 | 9983 | 0756 | 9971 | 40 |
| 21 | 0061 | 1.000 | 0236 | 9997 | 0410 | 9992 | 0584 | 9983 | 0758 | 9971 | 39 |
| 22 | 0064 | 1.000 | 0239 | 9997 | 0413 | 9991 | 0587 | 9983 | 0761 | 9971 | 38 |
| 23 | 0067 | 1.000 | 0241 | 9997 | 0416 | 9991 | 0590 | 9983 | 0764 | 9971 | 37 |
| 24 | 0070 | 1.000 | 0244 | 9997 | 0419 | 9991 | 0593 | 9982 | 0767 | 9971 | 36 |
| 25 | 0073 | 1.000 | 0247 | 9997 | 0422 | 9991 | 0596 | 9982 | 0770 | 9970 | 35 |
| 26 | 0076 | 1.000 | 0250 | 9997 | 0425 | 9991 | 0599 | 9982 | 0773 | 9970 | 34 |
| 27 | 0079 | 1.000 | 0253 | 9997 | 0427 | 9991 | 0602 | 9982 | 0776 | 9970 | 33 |
| 28 | 0081 | 1.000 | 0256 | 9997 | 0430 | 9991 | 0605 | 9982 | 0779 | 9970 | 32 |
| 29 | 0084 | 1.000 | 0259 | 9997 | 0433 | 9991 | 0608 | 9982 | 0782 | 9969 | 31 |
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| 43 | 0125 | 9999 | 0300 | 9996 | 0474 | 9989 | 0648 | 9979 | 0822 | 9966 | 17 |
| 44 | 0128 | 9999 | 0302 | 9995 | 0477 | 9989 | 0651 | 9979 | 0825 | 9966 | 16 |
| 45 | 0131 | 9999 | 0305 | 9995 | 0480 | 9988 | 0654 | 9979 | 0828 | 9966 | 15 |
| 46 | 0134 | 9999 | 0308 | 9995 | 0483 | 9988 | 0657 | 9978 | 0831 | 9965 | 14 |
| 47 | 0137 | 9999 | 0311 | 9995 | 0486 | 9988 | 0660 | 9978 | 0834 | 9965 | 13 |
| 48 | 0140 | 9999 | 0314 | 9995 | 0488 | 9988 | 0663 | 9978 | 0837 | 9965 | 12 |
| 49 | 0143 | 9999 | 0317 | 9995 | 0491 | 9988 | 0666 | 9978 | 0840 | 9965 | 11 |
| 50 | 0145 | 9999 | 0320 | 9995 | 0494 | 9988 | 0669 | 9978 | 0843 | 9964 | 10 |
| 51 | 0148 | 9999 | 0323 | 9995 | 0497 | 9988 | 0671 | 9977 | 0845 | 9964 | 9 |
| 52 | 0151 | 9999 | 0326 | 9995 | 0500 | 9987 | 0674 | 9977 | 0848 | 9964 | 8 |
| 53 | 0154 | 9999 | 0329 | 9995 | 0503 | 9987 | 0677 | 9977 | 0851 | 9964 | 7 |
| 54 | 0157 | 9999 | 0332 | 9995 | 0506 | 9987 | 0680 | 9977 | 0854 | 9963 | 6 |
| 55 | 0160 | 9999 | 0334 | 9994 | 0509 | 9987 | 0683 | 9977 | 0857 | 9963 | 5 |
| 56 | 0163 | 9999 | 0337 | 9994 | 0512 | 9987 | 0686 | 9976 | 0860 | 9963 | 4 |
| 57 | 0166 | 9999 | 0340 | 9994 | 0515 | 9987 | 0689 | 9976 | 0863 | 9963 | 3 |
| 58 | 0169 | 9999 | 0343 | 9994 | 0518 | 9987 | 0692 | 9976 | 0866 | 9962 | 2 |
| 59 | 0172 | 9999 | 0346 | 9994 | 0520 | 9986 | 0695 | 9976 | 0869 | 9962 | 1 |
| 60 | 0175 | 9999 | 0349 | 9994 | 0523 | 9986 | 0698 | 9976 | 0872 | 9962 | 0 |
| | cos | sin | cos | sin | cos | sin | cos | sin | cos | sin | |
| ′ | 89° | | 88° | | 87° | | 86° | | 85° | | ′ |

| ′ | 5° | | 6° | | 7° | | 8° | | 9° | | ′ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | sin | cos | sin | cos | sin | cos | sin | cos | sin | cos | |
| 0 | 0872 | 9962 | 1045 | 9945 | 1219 | 9925 | 1392 | 9903 | 1564 | 9877 | 60 |
| 1 | 0874 | 9962 | 1048 | 9945 | 1222 | 9925 | 1395 | 9902 | 1567 | 9876 | 59 |
| 2 | 0877 | 9961 | 1051 | 9945 | 1224 | 9925 | 1397 | 9902 | 1570 | 9876 | 58 |
| 3 | 0880 | 9961 | 1054 | 9944 | 1227 | 9924 | 1400 | 9901 | 1573 | 9876 | 57 |
| 4 | 0883 | 9961 | 1057 | 9944 | 1230 | 9924 | 1403 | 9901 | 1576 | 9875 | 56 |
| 5 | 0886 | 9961 | 1060 | 9944 | 1233 | 9924 | 1406 | 9901 | 1579 | 9875 | 55 |
| 6 | 0889 | 9960 | 1063 | 9943 | 1236 | 9923 | 1409 | 9900 | 1582 | 9874 | 54 |
| 7 | 0892 | 9960 | 1066 | 9943 | 1239 | 9923 | 1412 | 9900 | 1584 | 9874 | 53 |
| 8 | 0895 | 9960 | 1068 | 9943 | 1241 | 9923 | 1415 | 9899 | 1587 | 9873 | 52 |
| 9 | 0898 | 9960 | 1071 | 9942 | 1245 | 9922 | 1418 | 9899 | 1590 | 9873 | 51 |
| 10 | 0901 | 9959 | 1074 | 9942 | 1248 | 9922 | 1421 | 9899 | 1593 | 9872 | 50 |
| 11 | 0903 | 9959 | 1077 | 9942 | 1250 | 9922 | 1423 | 9898 | 1596 | 9872 | 49 |
| 12 | 0906 | 9959 | 1080 | 9942 | 1253 | 9921 | 1426 | 9898 | 1599 | 9871 | 48 |
| 13 | 0909 | 9959 | 1083 | 9941 | 1256 | 9921 | 1429 | 9897 | 1602 | 9871 | 47 |
| 14 | 0912 | 9958 | 1086 | 9941 | 1259 | 9920 | 1432 | 9897 | 1605 | 9870 | 46 |
| 15 | 0915 | 9958 | 1089 | 9941 | 1262 | 9920 | 1435 | 9897 | 1607 | 9870 | 45 |
| 16 | 0918 | 9958 | 1092 | 9940 | 1265 | 9920 | 1438 | 9896 | 1610 | 9869 | 44 |
| 17 | 0921 | 9958 | 1094 | 9940 | 1268 | 9919 | 1441 | 9896 | 1613 | 9869 | 43 |
| 18 | 0924 | 9957 | 1097 | 9940 | 1271 | 9919 | 1444 | 9895 | 1616 | 9869 | 42 |
| 19 | 0927 | 9957 | 1100 | 9939 | 1274 | 9919 | 1446 | 9895 | 1619 | 9868 | 41 |
| 20 | 0929 | 9957 | 1103 | 9939 | 1276 | 9918 | 1449 | 9894 | 1622 | 9868 | 40 |
| 21 | 0932 | 9956 | 1106 | 9939 | 1279 | 9918 | 1452 | 9894 | 1625 | 9867 | 39 |
| 22 | 0935 | 9956 | 1109 | 9938 | 1282 | 9917 | 1455 | 9894 | 1628 | 9867 | 38 |
| 23 | 0938 | 9956 | 1112 | 9938 | 1285 | 9917 | 1458 | 9893 | 1630 | 9866 | 37 |
| 24 | 0941 | 9956 | 1115 | 9938 | 1288 | 9917 | 1461 | 9893 | 1633 | 9866 | 36 |
| 25 | 0944 | 9955 | 1118 | 9937 | 1291 | 9916 | 1464 | 9802 | 1636 | 9865 | 35 |
| 26 | 0947 | 9955 | 1120 | 9937 | 1294 | 9916 | 1467 | 9892 | 1639 | 9865 | 34 |
| 27 | 0950 | 9955 | 1123 | 9937 | 1297 | 9916 | 1469 | 9891 | 1642 | 9864 | 33 |
| 28 | 0953 | 9955 | 1126 | 9936 | 1299 | 9915 | 1472 | 9891 | 1645 | 9864 | 32 |
| 29 | 0956 | 9954 | 1129 | 9936 | 1302 | 9915 | 1475 | 9891 | 1648 | 9863 | 31 |
| 30 | 0958 | 9954 | 1132 | 9936 | 1305 | 9914 | 1478 | 9890 | 1650 | 9863 | 30 |
| 31 | 0961 | 9954 | 1135 | 9935 | 1308 | 9914 | 1481 | 9890 | 1653 | 9862 | 29 |
| 32 | 0964 | 9953 | 1138 | 9935 | 1311 | 9914 | 1484 | 9889 | 1656 | 986 | 28 |
| 33 | 0967 | 9953 | 1141 | 9935 | 1314 | 9913 | 1487 | 9889 | 1659 | 986[illegible] | 27 |
| 34 | 0970 | 9953 | 1144 | 9934 | 1317 | 9913 | 1490 | 9888 | 1662 | 9861 | 26 |
| 35 | 0973 | 9953 | 1146 | 9934 | 1320 | 9913 | 1492 | 9888 | 1665 | 9860 | 25 |
| 36 | 0976 | 9952 | 1149 | 9934 | 1323 | 9912 | 1495 | 9888 | 1668 | 9860 | 24 |
| 37 | 0979 | 9952 | 1152 | 9933 | 1325 | 9912 | 1498 | 9887 | 1671 | 9859 | 23 |
| 38 | 0982 | 9952 | 1155 | 9933 | 1328 | 9911 | 1501 | 9887 | 1673 | 9859 | 22 |
| 39 | 0985 | 9951 | 1158 | 9933 | 1331 | 9911 | 1504 | 9886 | 1676 | 9859 | 21 |
| 40 | 0987 | 9951 | 1161 | 9932 | 1334 | 9911 | 1507 | 9886 | 1679 | 9858 | 20 |
| 41 | 0990 | 9951 | 1164 | 9932 | 1337 | 9910 | 1510 | 9885 | 1682 | 9858 | 19 |
| 42 | 0993 | 9951 | 1167 | 9932 | 1340 | 9910 | 1513 | 9885 | 1685 | 9857 | 18 |
| 43 | 0996 | 9950 | 1170 | 9931 | 1343 | 9909 | 1515 | 9884 | 1688 | 9857 | 17 |
| 44 | 0999 | 9950 | 1172 | 9931 | 1346 | 9909 | 1518 | 9884 | 1691 | 9856 | 16 |
| 45 | 1002 | 9950 | 1175 | 9931 | 1349 | 9909 | 1521 | 9884 | 1693 | 9856 | 15 |
| 46 | 1005 | 9949 | 1178 | 9930 | 1351 | 9908 | 1524 | 9883 | 1696 | 9855 | 14 |
| 47 | 1008 | 9949 | 1181 | 9930 | 1354 | 9908 | 1527 | 9883 | 1699 | 9855 | 13 |
| 48 | 1011 | 9949 | 1184 | 9930 | 1357 | 9907 | 1530 | 9882 | 1702 | 9854 | 12 |
| 49 | 1013 | 9949 | 1187 | 9929 | 1360 | 9907 | 1533 | 9882 | 1705 | 9854 | 11 |
| 50 | 1016 | 9948 | 1190 | 9929 | 1363 | 9907 | 1536 | 9881 | 1708 | 9853 | 10 |
| 51 | 1019 | 9948 | 1193 | 9929 | 1366 | 9906 | 1538 | 9881 | 1711 | 9853 | 9 |
| 52 | 1022 | 9948 | 1196 | 9928 | 1369 | 9906 | 1541 | 9880 | 1714 | 9852 | 8 |
| 53 | 1025 | 9947 | 1198 | 9928 | 1372 | 9905 | 1544 | 9880 | 1716 | 9852 | 7 |
| 54 | 1028 | 9947 | 1201 | 9928 | 1374 | 9905 | 1547 | 9880 | 1719 | 9851 | 6 |
| 55 | 1031 | 9947 | 1204 | 9927 | 1377 | 9905 | 1550 | 9879 | 1722 | 9851 | 5 |
| 56 | 1034 | 9946 | 1207 | 9927 | 1380 | 9904 | 1553 | 9879 | 1725 | 9850 | 4 |
| 57 | 1037 | 9946 | 1210 | 9927 | 1383 | 9904 | 1556 | 9878 | 1728 | 9850 | 3 |
| 58 | 1039 | 9946 | 1213 | 9926 | 1386 | 9903 | 1559 | 9878 | 1731 | 9849 | 2 |
| 59 | 1042 | 9946 | 1216 | 9926 | 1389 | 9903 | 1561 | 9877 | 1734 | 9849 | 1 |
| 60 | 1045 | 9945 | 1219 | 9925 | 1392 | 9903 | 1564 | 9877 | 1736 | 9848 | 0 |
| | cos | sin | cos | sin | cos | sin | cos | sin | cos | sin | |
| ′ | 84° | | 83° | | 82° | | 81° | | 80° | | ′ |

| ′ | 10° | | 11° | | 12° | | 13° | | 14° | | ′ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | sin | cos | sin | cos | sin | cos | sin | cos | sin | cos | |
| 0 | 1736 | 9848 | 1908 | 9816 | 2079 | 9781 | 2250 | 9744 | 2419 | 9703 | 60 |
| 1 | 1739 | 9848 | 1911 | 9816 | 2082 | 9781 | 2252 | 9743 | 2422 | 9702 | 59 |
| 2 | 1742 | 9847 | 1914 | 9815 | 2085 | 9780 | 2255 | 9742 | 2425 | 9702 | 58 |
| 3 | 1745 | 9847 | 1917 | 9815 | 2088 | 9780 | 2258 | 9742 | 2428 | 9701 | 57 |
| 4 | 1748 | 9846 | 1920 | 9814 | 2090 | 9779 | 2261 | 9741 | 2431 | 9700 | 56 |
| 5 | 1751 | 9846 | 1922 | 9813 | 2093 | 9778 | 2264 | 9740 | 2433 | 9699 | 55 |
| 6 | 1754 | 9845 | 1925 | 9813 | 2096 | 9778 | 2267 | 9740 | 2436 | 9699 | 54 |
| 7 | 1757 | 9845 | 1928 | 9812 | 2099 | 9777 | 2269 | 9739 | 2439 | 9698 | 53 |
| 8 | 1759 | 9844 | 1931 | 9812 | 2102 | 9777 | 2272 | 9738 | 2442 | 9697 | 52 |
| 9 | 1762 | 9843 | 1934 | 9811 | 2105 | 9776 | 2275 | 9738 | 2445 | 9697 | 51 |
| 10 | 1765 | 9843 | 1937 | 9811 | 2108 | 9775 | 2278 | 9737 | 2447 | 9696 | 50 |
| 11 | 1768 | 9842 | 1939 | 9810 | 2110 | 9775 | 2281 | 9736 | 2450 | 9695 | 49 |
| 12 | 1771 | 9842 | 1942 | 9810 | 2113 | 9774 | 2284 | 9736 | 2453 | 9694 | 48 |
| 13 | 1774 | 9841 | 1945 | 9809 | 2116 | 9774 | 2286 | 9735 | 2456 | 9694 | 47 |
| 14 | 1777 | 9841 | 1948 | 9808 | 2119 | 9773 | 2289 | 9734 | 2459 | 9693 | 46 |
| 15 | 1779 | 9840 | 1951 | 9808 | 2122 | 9772 | 2292 | 9734 | 2462 | 9692 | 45 |
| 16 | 1782 | 9840 | 1954 | 9807 | 2125 | 9772 | 2295 | 9733 | 2464 | 9692 | 44 |
| 17 | 1785 | 9839 | 1957 | 9807 | 2127 | 9771 | 2298 | 9732 | 2467 | 9691 | 43 |
| 18 | 1788 | 9839 | 1959 | 9806 | 2130 | 9770 | 2300 | 9732 | 2470 | 9690 | 42 |
| 19 | 1791 | 9838 | 1962 | 9806 | 2133 | 9770 | 2303 | 9731 | 2473 | 9689 | 41 |
| 20 | 1794 | 9838 | 1965 | 9805 | 2136 | 9769 | 2306 | 9730 | 2476 | 9689 | 40 |
| 21 | 1797 | 9837 | 1968 | 9804 | 2139 | 9769 | 2309 | 9730 | 2478 | 9688 | 39 |
| 22 | 1799 | 9837 | 1971 | 9804 | 2142 | 9768 | 2312 | 9729 | 2481 | 9687 | 38 |
| 23 | 1802 | 9836 | 1974 | 9803 | 2145 | 9767 | 2315 | 9728 | 2484 | 9687 | 37 |
| 24 | 1805 | 9836 | 1977 | 9803 | 2147 | 9767 | 2317 | 9728 | 2487 | 9686 | 36 |
| 25 | 1808 | 9835 | 1979 | 9802 | 2150 | 9766 | 2320 | 9727 | 2490 | 9685 | 35 |
| 26 | 1811 | 9835 | 1982 | 9802 | 2153 | 9765 | 2323 | 9726 | 2493 | 9684 | 34 |
| 27 | 1814 | 9834 | 1985 | 9801 | 2156 | 9765 | 2326 | 9726 | 2495 | 9684 | 33 |
| 28 | 1817 | 9834 | 1988 | 9800 | 2159 | 9764 | 2329 | 9725 | 2498 | 9683 | 32 |
| 29 | 1819 | 9833 | 1991 | 9800 | 2162 | 9764 | 2332 | 9724 | 2501 | 9682 | 31 |
| 30 | 1822 | 9833 | 1994 | 9799 | 2164 | 9763 | 2334 | 9724 | 2504 | 9681 | 30 |
| 31 | 1825 | 9832 | 1997 | 9799 | 2167 | 9762 | 2337 | 9723 | 2507 | 9681 | 29 |
| 32 | 1828 | 9831 | 1999 | 9798 | 2170 | 9762 | 2340 | 9722 | 2509 | 9680 | 28 |
| 33 | 1831 | 9831 | 2002 | 9798 | 2173 | 9761 | 2343 | 9722 | 2512 | 9679 | 27 |
| 34 | 1834 | 9830 | 2005 | 9797 | 2176 | 9760 | 2346 | 9721 | 2515 | 9679 | 26 |
| 35 | 1837 | 9830 | 2008 | 9796 | 2179 | 9760 | 2349 | 9720 | 2518 | 9678 | 25 |
| 36 | 1840 | 9829 | 2011 | 9796 | 2181 | 9759 | 2351 | 9720 | 2521 | 9677 | 24 |
| 37 | 1842 | 9829 | 2014 | 9795 | 2184 | 9759 | 2354 | 9719 | 2524 | 9676 | 23 |
| 38 | 1845 | 9828 | 2016 | 9795 | 2187 | 9758 | 2357 | 9718 | 2526 | 9676 | 22 |
| 39 | 1848 | 9828 | 2019 | 9794 | 2190 | 9757 | 2360 | 9718 | 2529 | 9675 | 21 |
| 40 | 1851 | 9827 | 2022 | 9793 | 2193 | 9757 | 2363 | 9717 | 2532 | 9674 | 20 |
| 41 | 1854 | 9827 | 2025 | 9793 | 2196 | 9756 | 2366 | 9716 | 2535 | 9673 | 19 |
| 42 | 1857 | 9826 | 2028 | 9792 | 2198 | 9755 | 2368 | 9715 | 2538 | 9673 | 18 |
| 43 | 1860 | 9826 | 2031 | 9792 | 2201 | 9755 | 2371 | 9715 | 2540 | 9672 | 17 |
| 44 | 1862 | 9825 | 2034 | 9791 | 2204 | 9754 | 2374 | 9714 | 2543 | 9671 | 16 |
| 45 | 1865 | 9825 | 2036 | 9790 | 2207 | 9753 | 2377 | 9713 | 2546 | 9670 | 15 |
| 46 | 1868 | 9824 | 2039 | 9790 | 2210 | 9753 | 2380 | 9713 | 2549 | 9670 | 14 |
| 47 | 1871 | 9823 | 2042 | 9789 | 2213 | 9752 | 2383 | 9712 | 2552 | 9669 | 13 |
| 48 | 1874 | 9823 | 2045 | 9789 | 2215 | 9751 | 2385 | 9711 | 2554 | 9668 | 12 |
| 49 | 1877 | 9822 | 2048 | 9788 | 2218 | 9751 | 2388 | 9711 | 2557 | 9667 | 11 |
| 50 | 1880 | 9822 | 2051 | 9787 | 2221 | 9750 | 2391 | 9710 | 2560 | 9667 | 10 |
| 51 | 1882 | 9821 | 2054 | 9787 | 2224 | 9750 | 2394 | 9709 | 2563 | 9666 | 9 |
| 52 | 1885 | 9821 | 2056 | 9786 | 2227 | 9749 | 2397 | 9709 | 2566 | 9665 | 8 |
| 53 | 1888 | 9820 | 2059 | 9786 | 2230 | 9748 | 2399 | 9708 | 2569 | 9665 | 7 |
| 54 | 1891 | 9820 | 2062 | 9785 | 2233 | 9748 | 2402 | 9707 | 2571 | 9664 | 6 |
| 55 | 1894 | 9819 | 2065 | 9784 | 2235 | 9747 | 2405 | 9706 | 2574 | 9663 | 5 |
| 56 | 1897 | 9818 | 2068 | 9784 | 2238 | 9746 | 2408 | 9706 | 2577 | 9662 | 4 |
| 57 | 1900 | 9818 | 2071 | 9783 | 2241 | 9746 | 2411 | 9705 | 2580 | 9662 | 3 |
| 58 | 1902 | 9817 | 2073 | 9783 | 2244 | 9745 | 2414 | 9704 | 2583 | 9661 | 2 |
| 59 | 1905 | 9817 | 2076 | 9782 | 2247 | 9744 | 2416 | 9704 | 2585 | 9660 | 1 |
| 60 | 1908 | 9816 | 2079 | 9781 | 2250 | 9744 | 2419 | 9703 | 2588 | 9659 | 0 |
| | cos | sin | cos | sin | cos | sin | cos | sin | cos | sin | |
| ′ | 79° | | 78° | | 77° | | 76° | | 75° | | ′ |

| ′ | 15° | | 16° | | 17° | | 18° | | 19° | | ′ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | sin | cos | sin | cos | sin | cos | sin | cos | sin | cos | |
| 0 | 2588 | 9659 | 2756 | 9613 | 2924 | 9563 | 3090 | 9511 | 3256 | 9455 | 60 |
| 1 | 2591 | 9659 | 2759 | 9612 | 2926 | 9562 | 3093 | 9510 | 3258 | 9454 | 59 |
| 2 | 2594 | 9658 | 2762 | 9611 | 2929 | 9561 | 3096 | 9509 | 3261 | 9453 | 58 |
| 3 | 2597 | 9657 | 2765 | 9610 | 2932 | 9560 | 3098 | 9508 | 3264 | 9452 | 57 |
| 4 | 2599 | 9656 | 2768 | 9609 | 2935 | 9560 | 3101 | 9507 | 3267 | 9451 | 56 |
| 5 | 2602 | 9655 | 2770 | 9609 | 2938 | 9559 | 3104 | 9506 | 3269 | 9450 | 55 |
| 6 | 2605 | 9655 | 2773 | 9608 | 2940 | 9558 | 3107 | 9505 | 3272 | 9449 | 54 |
| 7 | 2608 | 9654 | 2776 | 9607 | 2943 | 9557 | 3110 | 9504 | 3275 | 9449 | 53 |
| 8 | 2611 | 9653 | 2779 | 9606 | 2946 | 9556 | 3112 | 9503 | 3278 | 9448 | 52 |
| 9 | 2613 | 9652 | 2782 | 9605 | 2949 | 9555 | 3115 | 9502 | 3280 | 9447 | 51 |
| 10 | 2616 | 9652 | 2784 | 9605 | 2952 | 9555 | 3118 | 9502 | 3283 | 9446 | 50 |
| 11 | 2619 | 9651 | 2787 | 9604 | 2954 | 9554 | 3121 | 9501 | 3286 | 9445 | 49 |
| 12 | 2622 | 9650 | 2790 | 9603 | 2957 | 9553 | 3123 | 9500 | 3289 | 9444 | 48 |
| 13 | 2625 | 9649 | 2793 | 9602 | 2960 | 9552 | 3126 | 9499 | 3291 | 9443 | 47 |
| 14 | 2628 | 9649 | 2795 | 9601 | 2963 | 9551 | 3129 | 9498 | 3294 | 9442 | 46 |
| 15 | 2630 | 9648 | 2798 | 9600 | 2965 | 9550 | 3132 | 9497 | 3297 | 9441 | 45 |
| 16 | 2633 | 9647 | 2801 | 9600 | 2968 | 9549 | 3134 | 9496 | 3300 | 9440 | 44 |
| 17 | 2636 | 9646 | 2804 | 9599 | 2971 | 9548 | 3137 | 9495 | 3302 | 9439 | 43 |
| 18 | 2639 | 9646 | 2807 | 9598 | 2974 | 9548 | 3140 | 9494 | 3305 | 9438 | 42 |
| 19 | 2642 | 9645 | 2809 | 9597 | 2977 | 9547 | 3143 | 9493 | 3308 | 9437 | 41 |
| 20 | 2644 | 9644 | 2812 | 9596 | 2979 | 9546 | 3145 | 9492 | 3311 | 9436 | 40 |
| 21 | 2647 | 9643 | 2815 | 9596 | 2982 | 9545 | 3148 | 9492 | 3313 | 9435 | 39 |
| 22 | 2650 | 9642 | 2818 | 9595 | 2985 | 9544 | 3151 | 9491 | 3316 | 9434 | 38 |
| 23 | 2653 | 9642 | 2821 | 9594 | 2988 | 9543 | 3154 | 9490 | 3319 | 9433 | 37 |
| 24 | 2656 | 9641 | 2823 | 9593 | 2990 | 9542 | 3156 | 9489 | 3322 | 9432 | 36 |
| 25 | 2658 | 9640 | 2826 | 9592 | 2993 | 9542 | 3159 | 9488 | 3324 | 9431 | 35 |
| 26 | 2661 | 9639 | 2829 | 9591 | 2996 | 9541 | 3162 | 9487 | 3327 | 9430 | 34 |
| 27 | 2664 | 9639 | 2832 | 9591 | 2999 | 9540 | 3165 | 9486 | 3330 | 9429 | 33 |
| 28 | 2667 | 9638 | 2835 | 9590 | 3002 | 9539 | 3168 | 9485 | 3333 | 9428 | 32 |
| 29 | 2670 | 9637 | 2837 | 9589 | 3004 | 9538 | 3170 | 9484 | 3335 | 9427 | 31 |
| 30 | 2672 | 9636 | 2840 | 9588 | 3007 | 9537 | 3173 | 9483 | 3338 | 9426 | 30 |
| 31 | 2675 | 9636 | 2843 | 9587 | 3010 | 9536 | 3176 | 9482 | 3341 | 9425 | 29 |
| 32 | 2678 | 9635 | 2846 | 9587 | 3013 | 9535 | 3179 | 9481 | 3344 | 9424 | 28 |
| 33 | 2681 | 9634 | 2849 | 9586 | 3015 | 9535 | 3181 | 9480 | 3346 | 9423 | 27 |
| 34 | 2684 | 9633 | 2851 | 9585 | 3018 | 9534 | 3184 | 9480 | 3349 | 9423 | 26 |
| 35 | 2686 | 9632 | 2854 | 9584 | 3021 | 9533 | 3187 | 9479 | 3352 | 9422 | 25 |
| 36 | 2689 | 9632 | 2857 | 9583 | 3024 | 9532 | 3190 | 9478 | 3355 | 9421 | 24 |
| 37 | 2692 | 9631 | 2860 | 9582 | 3026 | 9531 | 3192 | 9477 | 3357 | 9420 | 23 |
| 38 | 2695 | 9630 | 2862 | 9582 | 3029 | 9530 | 3195 | 9476 | 3360 | 9419 | 22 |
| 39 | 2698 | 9629 | 2865 | 9581 | 3032 | 9529 | 3198 | 9475 | 3363 | 9418 | 21 |
| 40 | 2700 | 9628 | 2868 | 9580 | 3035 | 9528 | 3201 | 9474 | 3365 | 9417 | 20 |
| 41 | 2703 | 9628 | 2871 | 9579 | 3038 | 9527 | 3203 | 9473 | 3368 | 9416 | 19 |
| 42 | 2706 | 9627 | 2874 | 9578 | 3040 | 9527 | 3206 | 9472 | 3371 | 9415 | 18 |
| 43 | 2709 | 9626 | 2876 | 9577 | 3043 | 9526 | 3209 | 9471 | 3374 | 9414 | 17 |
| 44 | 2712 | 9625 | 2879 | 9577 | 3046 | 9525 | 3212 | 9470 | 3376 | 9413 | 16 |
| 45 | 2714 | 9625 | 2882 | 9576 | 3049 | 9524 | 3214 | 9469 | 3379 | 9412 | 15 |
| 46 | 2717 | 9624 | 2885 | 9575 | 3051 | 9523 | 3217 | 9468 | 3382 | 9411 | 14 |
| 47 | 2720 | 9623 | 2888 | 9574 | 3054 | 9522 | 3220 | 9467 | 3385 | 9410 | 13 |
| 48 | 2723 | 9622 | 2890 | 9573 | 3057 | 9521 | 3223 | 9466 | 3387 | 9409 | 12 |
| 49 | 2726 | 9621 | 2893 | 9572 | 3060 | 9520 | 3225 | 9466 | 3390 | 9408 | 11 |
| 50 | 2728 | 9621 | 2896 | 9572 | 3062 | 9520 | 3228 | 9465 | 3393 | 9407 | 10 |
| 51 | 2731 | 9620 | 2899 | 9571 | 3065 | 9519 | 3231 | 9464 | 3396 | 9406 | 9 |
| 52 | 2734 | 9619 | 2901 | 9570 | 3068 | 9518 | 3234 | 9463 | 3398 | 9405 | 8 |
| 53 | 2737 | 9618 | 2904 | 9569 | 3071 | 9517 | 3236 | 9462 | 3401 | 9404 | 7 |
| 54 | 2740 | 9617 | 2907 | 9568 | 3074 | 9516 | 3239 | 9461 | 3404 | 9403 | 6 |
| 55 | 2742 | 9617 | 2910 | 9567 | 3076 | 9515 | 3242 | 9460 | 3407 | 9402 | 5 |
| 56 | 2745 | 9616 | 2913 | 9566 | 3079 | 9514 | 3245 | 9459 | 3409 | 9401 | 4 |
| 57 | 2748 | 9615 | 2915 | 9566 | 3082 | 9513 | 3247 | 9458 | 3412 | 9400 | 3 |
| 58 | 2751 | 9614 | 2918 | 9565 | 3085 | 9512 | 3250 | 9457 | 3415 | 9399 | 2 |
| 59 | 2754 | 9613 | 2921 | 9564 | 3087 | 9511 | 3253 | 9456 | 3417 | 9398 | 1 |
| 60 | 2756 | 9613 | 2924 | 9563 | 3090 | 9511 | 3256 | 9455 | 3420 | 9397 | 0 |
| | cos | sin | cos | sin | cos | sin | cos | sin | cos | sin | |
| ′ | 74° | | 73° | | 72° | | 71° | | 70° | | ′ |

| ′ | 20° sin | 20° cos | 21° sin | 21° cos | 22° sin | 22° cos | 23° sin | 23° cos | 24° sin | 24° cos | ′ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 3420 | 9397 | 3584 | 9336 | 3746 | 9272 | 3907 | 9205 | 4067 | 9135 | 60 |
| 1 | 3423 | 9396 | 3586 | 9335 | 3749 | 9271 | 3910 | 9204 | 4070 | 9134 | 59 |
| 2 | 3426 | 9395 | 3589 | 9334 | 3751 | 9270 | 3913 | 9203 | 4073 | 9133 | 58 |
| 3 | 3428 | 9394 | 3592 | 9333 | 3754 | 9269 | 3915 | 9202 | 4075 | 9132 | 57 |
| 4 | 3431 | 9393 | 3595 | 9332 | 3757 | 9267 | 3918 | 9200 | 4078 | 9131 | 56 |
| 5 | 3434 | 9392 | 3597 | 9331 | 3760 | 9266 | 3921 | 9199 | 4081 | 9130 | 55 |
| 6 | 3437 | 9391 | 3600 | 9330 | 3762 | 9265 | 3923 | 9198 | 4083 | 9128 | 54 |
| 7 | 3439 | 9390 | 3603 | 9328 | 3765 | 9264 | 3926 | 9197 | 4086 | 9127 | 53 |
| 8 | 3442 | 9389 | 3605 | 9327 | 3768 | 9263 | 3929 | 9196 | 4089 | 9126 | 52 |
| 9 | 3445 | 9388 | 3608 | 9326 | 3770 | 9262 | 3931 | 9195 | 4091 | 9125 | 51 |
| 10 | 3448 | 9387 | 3611 | 9325 | 3773 | 9261 | 3934 | 9194 | 4094 | 9124 | 50 |
| 11 | 3450 | 9386 | 3614 | 9324 | 3776 | 9260 | 3937 | 9192 | 4097 | 9122 | 49 |
| 12 | 3453 | 9385 | 3616 | 9323 | 3778 | 9259 | 3939 | 9191 | 4099 | 9121 | 48 |
| 13 | 3456 | 9384 | 3619 | 9322 | 3781 | 9258 | 3942 | 9190 | 4102 | 9120 | 47 |
| 14 | 3458 | 9383 | 3622 | 9321 | 3784 | 9257 | 3945 | 9189 | 4105 | 9119 | 46 |
| 15 | 3461 | 9382 | 3624 | 9320 | 3786 | 9255 | 3947 | 9188 | 4107 | 9118 | 45 |
| 16 | 3464 | 9381 | 3627 | 9319 | 3789 | 9254 | 3950 | 9187 | 4110 | 9116 | 44 |
| 17 | 3467 | 9380 | 3630 | 9318 | 3792 | 9253 | 3953 | 9186 | 4112 | 9115 | 43 |
| 18 | 3469 | 9379 | 3633 | 9317 | 3795 | 9252 | 3955 | 9184 | 4115 | 9114 | 42 |
| 19 | 3472 | 9378 | 3635 | 9316 | 3797 | 9251 | 3958 | 9183 | 4118 | 9113 | 41 |
| 20 | 3475 | 9377 | 3638 | 9315 | 3800 | 9250 | 3961 | 9182 | 4120 | 9112 | 40 |
| 21 | 3478 | 9376 | 3641 | 9314 | 3803 | 9249 | 3963 | 9181 | 4123 | 9110 | 39 |
| 22 | 3480 | 9375 | 3643 | 9313 | 3805 | 9248 | 3966 | 9180 | 4126 | 9109 | 38 |
| 23 | 3483 | 9374 | 3646 | 9312 | 3808 | 9247 | 3969 | 9179 | 4128 | 9108 | 37 |
| 24 | 3486 | 9373 | 3649 | 9311 | 3811 | 9245 | 3971 | 9178 | 4131 | 9107 | 36 |
| 25 | 3488 | 9372 | 3651 | 9309 | 3813 | 9244 | 3974 | 9176 | 4134 | 9106 | 35 |
| 26 | 3491 | 9371 | 3654 | 9308 | 3816 | 9243 | 3977 | 9175 | 4136 | 9104 | 34 |
| 27 | 3494 | 9370 | 3657 | 9307 | 3819 | 9242 | 3979 | 9174 | 4139 | 9103 | 33 |
| 28 | 3497 | 9369 | 3660 | 9306 | 3821 | 9241 | 3982 | 9173 | 4142 | 9102 | 32 |
| 29 | 3499 | 9368 | 3662 | 9305 | 3824 | 9240 | 3985 | 9172 | 4144 | 9101 | 31 |
| 30 | 3502 | 9367 | 3665 | 9304 | 3827 | 9239 | 3987 | 9171 | 4147 | 9100 | 30 |
| 31 | 3505 | 9366 | 3668 | 9303 | 3830 | 9238 | 3990 | 9169 | 4150 | 9098 | 29 |
| 32 | 3508 | 9365 | 3670 | 9302 | 3832 | 9237 | 3993 | 9168 | 4152 | 9097 | 28 |
| 33 | 3510 | 9364 | 3673 | 9301 | 3835 | 9235 | 3995 | 9167 | 4155 | 9096 | 27 |
| 34 | 3513 | 9363 | 3676 | 9300 | 3838 | 9234 | 3998 | 9166 | 4158 | 9095 | 26 |
| 35 | 3516 | 9362 | 3679 | 9299 | 3840 | 9233 | 4001 | 9165 | 4160 | 9094 | 25 |
| 36 | 3518 | 9361 | 3681 | 9298 | 3843 | 9232 | 4003 | 9164 | 4163 | 9092 | 24 |
| 37 | 3521 | 9360 | 3684 | 9297 | 3846 | 9231 | 4006 | 9162 | 4165 | 9091 | 23 |
| 38 | 3524 | 9359 | 3687 | 9296 | 3848 | 9230 | 4009 | 9161 | 4168 | 9090 | 22 |
| 39 | 3527 | 9358 | 3689 | 9295 | 3851 | 9229 | 4011 | 9160 | 4171 | 9089 | 21 |
| 40 | 3529 | 9356 | 3692 | 9293 | 3854 | 9228 | 4014 | 9159 | 4173 | 9088 | 20 |
| 41 | 3532 | 9355 | 3695 | 9292 | 3856 | 9227 | 4017 | 9158 | 4176 | 9086 | 19 |
| 42 | 3535 | 9354 | 3697 | 9291 | 3859 | 9225 | 4019 | 9157 | 4179 | 9085 | 18 |
| 43 | 3537 | 9353 | 3700 | 9290 | 3862 | 9224 | 4022 | 9155 | 4181 | 9084 | 17 |
| 44 | 3540 | 9352 | 3703 | 9289 | 3864 | 9223 | 4025 | 9154 | 4184 | 9083 | 16 |
| 45 | 3543 | 9351 | 3706 | 9288 | 3867 | 9222 | 4027 | 9153 | 4187 | 9081 | 15 |
| 46 | 3546 | 9350 | 3708 | 9287 | 3870 | 9221 | 4030 | 9152 | 4189 | 9080 | 14 |
| 47 | 3548 | 9349 | 3711 | 9286 | 3872 | 9220 | 4033 | 9151 | 4192 | 9079 | 13 |
| 48 | 3551 | 9348 | 3714 | 9285 | 3875 | 9219 | 4035 | 9150 | 4195 | 9078 | 12 |
| 49 | 3554 | 9347 | 3716 | 9284 | 3878 | 9218 | 4038 | 9148 | 4197 | 9077 | 11 |
| 50 | 3557 | 9346 | 3719 | 9283 | 3881 | 9216 | 4041 | 9147 | 4200 | 9075 | 10 |
| 51 | 3559 | 9345 | 3722 | 9282 | 3883 | 9215 | 4043 | 9146 | 4202 | 9074 | 9 |
| 52 | 3562 | 9344 | 3724 | 9281 | 3886 | 9214 | 4046 | 9145 | 4205 | 9073 | 8 |
| 53 | 3565 | 9343 | 3727 | 9279 | 3889 | 9213 | 4049 | 9144 | 4208 | 9072 | 7 |
| 54 | 3567 | 9342 | 3730 | 9278 | 3891 | 9212 | 4051 | 9143 | 4210 | 9070 | 6 |
| 55 | 3570 | 9341 | 3733 | 9277 | 3894 | 9211 | 4054 | 9141 | 4213 | 9069 | 5 |
| 56 | 3573 | 9340 | 3735 | 9276 | 3897 | 9210 | 4057 | 9140 | 4216 | 9068 | 4 |
| 57 | 3576 | 9339 | 3738 | 9275 | 3899 | 9208 | 4059 | 9139 | 4218 | 9067 | 3 |
| 58 | 3578 | 9338 | 3741 | 9274 | 3902 | 9207 | 4062 | 9138 | 4221 | 9066 | 2 |
| 59 | 3581 | 9337 | 3743 | 9273 | 3905 | 9206 | 4065 | 9137 | 4224 | 9064 | 1 |
| 60 | 3584 | 9336 | 3746 | 9272 | 3907 | 9205 | 4067 | 9135 | 4226 | 9063 | 0 |
| | cos | sin | cos | sin | cos | sin | cos | sin | cos | sin | |
| ′ | 69° | | 68° | | 67° | | 66° | | 65° | | ′ |

| ′ | 25° | | 26° | | 27° | | 28° | | 29° | | ′ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | sin | cos | sin | cos | sin | cos | sin | cos | sin | cos | |
| 0 | 4226 | 9063 | 4384 | 8988 | 4540 | 8910 | 4695 | 8829 | 4848 | 8746 | 60 |
| 1 | 4229 | 9062 | 4386 | 8987 | 4542 | 8909 | 4697 | 8828 | 4851 | 8745 | 59 |
| 2 | 4231 | 9061 | 4389 | 8985 | 4545 | 8907 | 4700 | 8827 | 4853 | 8743 | 58 |
| 3 | 4234 | 9059 | 4392 | 8984 | 4548 | 8906 | 4702 | 8825 | 4856 | 8742 | 57 |
| 4 | 4237 | 9058 | 4394 | 8983 | 4550 | 8905 | 4705 | 8824 | 4858 | 8741 | 56 |
| 5 | 4239 | 9057 | 4397 | 8982 | 4553 | 8903 | 4708 | 8823 | 4861 | 8739 | 55 |
| 6 | 4242 | 9056 | 4399 | 8980 | 4555 | 8902 | 4710 | 8821 | 4863 | 8738 | 54 |
| 7 | 4245 | 9054 | 4402 | 8979 | 4558 | 8901 | 4713 | 8820 | 4866 | 8736 | 53 |
| 8 | 4247 | 9053 | 4405 | 8978 | 4561 | 8899 | 4715 | 8819 | 4868 | 8735 | 52 |
| 9 | 4250 | 9052 | 4407 | 8976 | 4563 | 8898 | 4718 | 8817 | 4871 | 8733 | 51 |
| 10 | 4253 | 9051 | 4410 | 8975 | 4566 | 8897 | 4720 | 8816 | 4874 | 8732 | 50 |
| 11 | 4255 | 9050 | 4412 | 8974 | 4568 | 8895 | 4723 | 8814 | 4876 | 8731 | 49 |
| 12 | 4258 | 9048 | 4415 | 8973 | 4571 | 8894 | 4726 | 8813 | 4879 | 8729 | 48 |
| 13 | 4260 | 9047 | 4418 | 8971 | 4574 | 8893 | 4728 | 8812 | 4881 | 8728 | 47 |
| 14 | 4263 | 9046 | 4420 | 8970 | 4576 | 8892 | 4731 | 8810 | 4884 | 8726 | 46 |
| 15 | 4266 | 9045 | 4423 | 8969 | 4579 | 8890 | 4733 | 8809 | 4886 | 8725 | 45 |
| 16 | 4268 | 9043 | 4425 | 8967 | 4581 | 8889 | 4736 | 8808 | 4889 | 8724 | 44 |
| 17 | 4271 | 9042 | 4428 | 8966 | 4584 | 8888 | 4738 | 8806 | 4891 | 8722 | 43 |
| 18 | 4274 | 9041 | 4431 | 8965 | 4586 | 8886 | 4741 | 8805 | 4894 | 8721 | 42 |
| 19 | 4276 | 9040 | 4433 | 8964 | 4589 | 8885 | 4743 | 8803 | 4896 | 8719 | 41 |
| 20 | 4279 | 9038 | 4436 | 8962 | 4592 | 8884 | 4746 | 8802 | 4899 | 8718 | 40 |
| 21 | 4281 | 9037 | 4439 | 8961 | 4594 | 8882 | 4749 | 8801 | 4901 | 8716 | 39 |
| 22 | 4284 | 9036 | 4441 | 8960 | 4597 | 8881 | 4751 | 8799 | 4904 | 8715 | 38 |
| 23 | 4287 | 9035 | 4444 | 8958 | 4599 | 8879 | 4754 | 8798 | 4907 | 8714 | 37 |
| 24 | 4289 | 9033 | 4446 | 8957 | 4602 | 8878 | 4756 | 8796 | 4909 | 8712 | 36 |
| 25 | 4292 | 9032 | 4449 | 8956 | 4605 | 8877 | 4759 | 8795 | 4912 | 8711 | 35 |
| 26 | 4295 | 9031 | 4452 | 8955 | 4607 | 8875 | 4761 | 8794 | 4914 | 8709 | 34 |
| 27 | 4297 | 9030 | 4454 | 8953 | 4610 | 8874 | 4764 | 8792 | 4917 | 8708 | 33 |
| 28 | 4300 | 9028 | 4457 | 8952 | 4612 | 8873 | 4766 | 8791 | 4919 | 8706 | 32 |
| 29 | 4302 | 9027 | 4459 | 8951 | 4615 | 8871 | 4769 | 8790 | 4922 | 8705 | 31 |
| 30 | 4305 | 9026 | 4462 | 8949 | 4617 | 8870 | 4772 | 8788 | 4924 | 8704 | 30 |
| 31 | 4308 | 9025 | 4465 | 8948 | 4620 | 8869 | 4774 | 8787 | 4927 | 8702 | 29 |
| 32 | 4310 | 9023 | 4467 | 8947 | 4623 | 8867 | 4777 | 8785 | 4929 | 8701 | 28 |
| 33 | 4313 | 9022 | 4470 | 8945 | 4625 | 8866 | 4779 | 8784 | 4932 | 8699 | 27 |
| 34 | 4316 | 9021 | 4472 | 8944 | 4628 | 8865 | 4782 | 8783 | 4934 | 8698 | 26 |
| 35 | 4318 | 9020 | 4475 | 8943 | 4630 | 8863 | 4784 | 8781 | 4937 | 8696 | 25 |
| 36 | 4321 | 9018 | 4478 | 8942 | 4633 | 8862 | 4787 | 8780 | 4939 | 8695 | 24 |
| 37 | 4323 | 9017 | 4480 | 8940 | 4636 | 8861 | 4789 | 8778 | 4942 | 8694 | 23 |
| 38 | 4326 | 9016 | 4483 | 8939 | 4638 | 8859 | 4792 | 8777 | 4944 | 8692 | 22 |
| 39 | 4329 | 9015 | 4485 | 8938 | 4641 | 8858 | 4795 | 8776 | 4947 | 8691 | 21 |
| 40 | 4331 | 9013 | 4488 | 8936 | 4643 | 8857 | 4797 | 8774 | 4950 | 8689 | 20 |
| 41 | 4334 | 9012 | 4491 | 8935 | 4646 | 8855 | 4800 | 8773 | 4952 | 8688 | 19 |
| 42 | 4337 | 9011 | 4493 | 8934 | 4648 | 8854 | 4802 | 8771 | 4955 | 8686 | 18 |
| 43 | 4339 | 9010 | 4496 | 8932 | 4651 | 8853 | 4805 | 8770 | 4957 | 8685 | 17 |
| 44 | 4342 | 9008 | 4498 | 8931 | 4654 | 8851 | 4807 | 8769 | 4960 | 8683 | 16 |
| 45 | 4344 | 9007 | 4501 | 8930 | 4656 | 8850 | 4810 | 8767 | 4962 | 8682 | 15 |
| 46 | 4347 | 9006 | 4504 | 8928 | 4659 | 8849 | 4812 | 8766 | 4965 | 8681 | 14 |
| 47 | 4350 | 9004 | 4506 | 8927 | 4661 | 8847 | 4815 | 8764 | 4967 | 8679 | 13 |
| 48 | 4352 | 9003 | 4509 | 8926 | 4664 | 8846 | 4818 | 8763 | 4970 | 8678 | 12 |
| 49 | 4355 | 9002 | 4511 | 8925 | 4666 | 8844 | 4820 | 8762 | 4972 | 8676 | 11 |
| 50 | 4358 | 9001 | 4514 | 8928 | 4669 | 8843 | 4823 | 8760 | 4975 | 8675 | 10 |
| 51 | 4360 | 8999 | 4517 | 8922 | 4672 | 8842 | 4825 | 8759 | 4977 | 8673 | 9 |
| 52 | 4363 | 8998 | 4519 | 8921 | 4674 | 8840 | 4828 | 8757 | 4980 | 8672 | 8 |
| 53 | 4365 | 8997 | 4522 | 8919 | 4677 | 8839 | 4830 | 8756 | 4982 | 8670 | 7 |
| 54 | 4368 | 8996 | 4524 | 8918 | 4679 | 8838 | 4833 | 8755 | 4985 | 8669 | 6 |
| 55 | 4371 | 8994 | 4527 | 8917 | 4682 | 8836 | 4835 | 8753 | 4987 | 8668 | 5 |
| 56 | 4373 | 8993 | 4530 | 8915 | 4684 | 8835 | 4838 | 8752 | 4990 | 8666 | 4 |
| 57 | 4376 | 8992 | 4532 | 8914 | 4687 | 8834 | 4840 | 8750 | 4992 | 8665 | 3 |
| 58 | 4378 | 8990 | 4535 | 8913 | 4690 | 8832 | 4843 | 8749 | 4995 | 8663 | 2 |
| 59 | 4381 | 8989 | 4537 | 8911 | 4692 | 8831 | 4846 | 8748 | 4997 | 8662 | 1 |
| 60 | 4384 | 8988 | 4540 | 8910 | 4695 | 8829 | 4848 | 8746 | 5000 | 8660 | 0 |
| | cos | sin | cos | sin | cos | sin | cos | sin | cos | sin | |
| ′ | 64° | | 63° | | 62° | | 61° | | 60° | | ′ |

| ′ | 30° | | 31° | | 32° | | 33° | | 34° | | ′ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | sin | cos | sin | cos | sin | cos | sin | cos | sin | cos | |
| 0 | 5000 | 8660 | 5150 | 8572 | 5299 | 8480 | 5446 | 8387 | 5592 | 8290 | 60 |
| 1 | 5003 | 8659 | 5153 | 8570 | 5302 | 8479 | 5449 | 8385 | 5594 | 8289 | 59 |
| 2 | 5005 | 8657 | 5155 | 8569 | 5304 | 8477 | 5451 | 8384 | 5597 | 8287 | 58 |
| 3 | 5008 | 8656 | 5158 | 8567 | 5307 | 8476 | 5454 | 8382 | 5599 | 8285 | 57 |
| 4 | 5010 | 8654 | 5160 | 8566 | 5309 | 8474 | 5456 | 8380 | 5602 | 8284 | 56 |
| 5 | 5013 | 8653 | 5163 | 8564 | 5312 | 8473 | 5459 | 8379 | 5604 | 8282 | 55 |
| 6 | 5015 | 8652 | 5165 | 8563 | 5314 | 8471 | 5461 | 8377 | 5606 | 8281 | 54 |
| 7 | 5018 | 8650 | 5168 | 8561 | 5316 | 8470 | 5463 | 8376 | 5609 | 8279 | 53 |
| 8 | 5020 | 8649 | 5170 | 8560 | 5319 | 8468 | 5466 | 8374 | 5611 | 8277 | 52 |
| 9 | 5023 | 8647 | 5173 | 8558 | 5321 | 8467 | 5468 | 8372 | 5614 | 8276 | 51 |
| 10 | 5025 | 8646 | 5175 | 8557 | 5324 | 8465 | 5471 | 8371 | 5616 | 8274 | 50 |
| 11 | 5028 | 8644 | 5178 | 8555 | 5326 | 8463 | 5473 | 8369 | 5618 | 8272 | 49 |
| 12 | 5030 | 8643 | 5180 | 8554 | 5329 | 8462 | 5476 | 8368 | 5621 | 8271 | 48 |
| 13 | 5033 | 8641 | 5183 | 8552 | 5331 | 8460 | 5478 | 8366 | 5623 | 8269 | 47 |
| 14 | 5035 | 8640 | 5185 | 8551 | 5334 | 8459 | 5480 | 8364 | 5626 | 8268 | 46 |
| 15 | 5038 | 8638 | 5188 | 8549 | 5336 | 8457 | 5483 | 8363 | 5628 | 8266 | 45 |
| 16 | 5040 | 8637 | 5190 | 8548 | 5339 | 8456 | 5485 | 8361 | 5630 | 8264 | 44 |
| 17 | 5043 | 8635 | 5193 | 8546 | 5341 | 8454 | 5488 | 8360 | 5633 | 8263 | 43 |
| 18 | 5045 | 8634 | 5195 | 8545 | 5344 | 8453 | 5490 | 8358 | 5635 | 8261 | 42 |
| 19 | 5048 | 8632 | 5198 | 8543 | 5346 | 8451 | 5493 | 8356 | 5638 | 8259 | 41 |
| 20 | 5050 | 8631 | 5200 | 8542 | 5348 | 8450 | 5495 | 8355 | 5640 | 8258 | 40 |
| 21 | 5053 | 8630 | 5203 | 8540 | 5351 | 8448 | 5498 | 8353 | 5642 | 8256 | 39 |
| 22 | 5055 | 8628 | 5205 | 8539 | 5353 | 8446 | 5500 | 8352 | 5645 | 8254 | 38 |
| 23 | 5058 | 8627 | 5208 | 8537 | 5356 | 8445 | 5502 | 8350 | 5647 | 8253 | 37 |
| 24 | 5060 | 8625 | 5210 | 8536 | 5358 | 8443 | 5505 | 8348 | 5650 | 8251 | 36 |
| 25 | 5063 | 8624 | 5213 | 8534 | 5361 | 8442 | 5507 | 8347 | 5652 | 8249 | 35 |
| 26 | 5065 | 8622 | 5215 | 8532 | 5363 | 8440 | 5510 | 8345 | 5654 | 8248 | 34 |
| 27 | 5068 | 8621 | 5218 | 8531 | 5366 | 8439 | 5512 | 8344 | 5657 | 8246 | 33 |
| 28 | 5070 | 8619 | 5220 | 8529 | 5368 | 8437 | 5515 | 8342 | 5659 | 8245 | 32 |
| 29 | 5073 | 8618 | 5223 | 8528 | 5371 | 8435 | 5517 | 8340 | 5662 | 8243 | 31 |
| 30 | 5075 | 8616 | 5225 | 8526 | 5373 | 8434 | 5519 | 8339 | 5664 | 8241 | 30 |
| 31 | 5078 | 8615 | 5227 | 8525 | 5375 | 8432 | 5522 | 8337 | 5666 | 8240 | 29 |
| 32 | 5080 | 8613 | 5230 | 8523 | 5378 | 8431 | 5524 | 8336 | 5669 | 8238 | 28 |
| 33 | 5083 | 8612 | 5232 | 8522 | 5380 | 8429 | 5527 | 8334 | 5671 | 8236 | 27 |
| 34 | 5085 | 8610 | 5235 | 8520 | 5383 | 8428 | 5529 | 8332 | 5674 | 8235 | 26 |
| 35 | 5088 | 8609 | 5237 | 8519 | 5385 | 8426 | 5531 | 8331 | 5676 | 8233 | 25 |
| 36 | 5090 | 8607 | 5240 | 8517 | 5388 | 8425 | 5534 | 8329 | 5678 | 8231 | 24 |
| 37 | 5093 | 8606 | 5242 | 8516 | 5390 | 8423 | 5536 | 8328 | 5681 | 8230 | 23 |
| 38 | 5095 | 8604 | 5245 | 8514 | 5393 | 8421 | 5539 | 8326 | 5683 | 8228 | 22 |
| 39 | 5098 | 8603 | 5247 | 8513 | 5395 | 8420 | 5541 | 8324 | 5686 | 8226 | 21 |
| 40 | 5100 | 8601 | 5250 | 8511 | 5398 | 8418 | 5544 | 8323 | 5688 | 8225 | 20 |
| 41 | 5103 | 8600 | 5252 | 8510 | 5400 | 8417 | 5546 | 8321 | 5690 | 8223 | 19 |
| 42 | 5105 | 8599 | 5255 | 8508 | 5402 | 8415 | 5548 | 8320 | 5693 | 8221 | 18 |
| 43 | 5108 | 8597 | 5257 | 8507 | 5405 | 8414 | 5551 | 8318 | 5695 | 8220 | 17 |
| 44 | 5110 | 8596 | 5260 | 8505 | 5407 | 8412 | 5553 | 8316 | 5698 | 8218 | 16 |
| 45 | 5113 | 8594 | 5262 | 8504 | 5410 | 8410 | 5556 | 8315 | 5700 | 8216 | 15 |
| 46 | 5115 | 8593 | 5265 | 8502 | 5412 | 8409 | 5558 | 8313 | 5702 | 8215 | 14 |
| 47 | 5118 | 8591 | 5267 | 8500 | 5415 | 8407 | 5561 | 8311 | 5705 | 8213 | 13 |
| 48 | 5120 | 8590 | 5270 | 8499 | 5417 | 8406 | 5563 | 8310 | 5707 | 8211 | 12 |
| 49 | 5123 | 8588 | 5272 | 8497 | 5420 | 8404 | 5565 | 8308 | 5710 | 8210 | 11 |
| 50 | 5125 | 8587 | 5275 | 8496 | 5422 | 8403 | 5568 | 8307 | 5712 | 8208 | 10 |
| 51 | 5128 | 8585 | 5277 | 8494 | 5424 | 8401 | 5570 | 8305 | 5714 | 8207 | 9 |
| 52 | 5130 | 8584 | 5279 | 8493 | 5427 | 8399 | 5573 | 8303 | 5717 | 8205 | 8 |
| 53 | 5133 | 8582 | 5282 | 8491 | 5429 | 8398 | 5575 | 8302 | 5719 | 8203 | 7 |
| 54 | 5135 | 8581 | 5284 | 8490 | 5432 | 8396 | 5577 | 8300 | 5721 | 8202 | 6 |
| 55 | 5138 | 8579 | 5287 | 8488 | 5434 | 8395 | 5580 | 8299 | 5724 | 8200 | 5 |
| 56 | 5140 | 8578 | 5289 | 8487 | 5437 | 8393 | 5582 | 8297 | 5726 | 8198 | 4 |
| 57 | 5143 | 8576 | 5292 | 8485 | 5439 | 8391 | 5585 | 8295 | 5729 | 8197 | 3 |
| 58 | 5145 | 8575 | 5294 | 8484 | 5442 | 8390 | 5587 | 8294 | 5731 | 8195 | 2 |
| 59 | 5148 | 8573 | 5297 | 8482 | 5444 | 8388 | 5590 | 8292 | 5733 | 8193 | 1 |
| 60 | 5150 | 8572 | 5299 | 8480 | 5446 | 8387 | 5592 | 8290 | 5736 | 8192 | 0 |
| | cos | sin | cos | sin | cos | sin | cos | sin | cos | sin | |
| ′ | 59° | | 58° | | 57° | | 56° | | 55° | | ′ |

| ′ | 35° | | 36° | | 37° | | 38° | | 39° | | ′ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | sin | cos | sin | cos | sin | cos | sin | cos | sin | cos | |
| 0 | 5736 | 8192 | 5878 | 8090 | 6018 | 7986 | 6157 | 7880 | 6293 | 7771 | 60 |
| 1 | 5738 | 8190 | 5880 | 8088 | 6020 | 7985 | 6159 | 7878 | 6295 | 7770 | 59 |
| 2 | 5741 | 8188 | 5883 | 8087 | 6023 | 7983 | 6161 | 7877 | 6298 | 7768 | 58 |
| 3 | 5743 | 8187 | 5885 | 8085 | 6025 | 7981 | 6163 | 7875 | 6300 | 7766 | 57 |
| 4 | 5745 | 8185 | 5887 | 8083 | 6027 | 7979 | 6166 | 7873 | 6302 | 7764 | 56 |
| 5 | 5748 | 8183 | 5890 | 8082 | 6030 | 7978 | 6168 | 7871 | 6305 | 7762 | 55 |
| 6 | 5750 | 8181 | 5892 | 8080 | 6032 | 7976 | 6170 | 7869 | 6307 | 7760 | 54 |
| 7 | 5752 | 8180 | 5894 | 8078 | 6034 | 7974 | 6173 | 7868 | 6309 | 7759 | 53 |
| 8 | 5755 | 8178 | 5897 | 8076 | 6037 | 7972 | 6175 | 7866 | 6311 | 7757 | 52 |
| 9 | 5757 | 8176 | 5899 | 8075 | 6039 | 7971 | 6177 | 7864 | 6314 | 7755 | 51 |
| 10 | 5760 | 8175 | 5901 | 8073 | 6041 | 7969 | 6180 | 7862 | 6316 | 7753 | 50 |
| 11 | 5762 | 8173 | 5904 | 8071 | 6044 | 7967 | 6182 | 7860 | 6318 | 7751 | 49 |
| 12 | 5764 | 8171 | 5906 | 8070 | 6046 | 7965 | 6184 | 7859 | 6320 | 7749 | 48 |
| 13 | 5767 | 8170 | 5908 | 8068 | 6048 | 7964 | 6186 | 7857 | 6323 | 7748 | 47 |
| 14 | 5769 | 8168 | 5911 | 8066 | 6051 | 7962 | 6189 | 7855 | 6325 | 7746 | 46 |
| 15 | 5771 | 8166 | 5913 | 8064 | 6053 | 7960 | 6191 | 7853 | 6327 | 7744 | 45 |
| 16 | 5774 | 8165 | 5915 | 8063 | 6055 | 7958 | 6193 | 7851 | 6329 | 7742 | 44 |
| 17 | 5776 | 8163 | 5918 | 8061 | 6058 | 7956 | 6196 | 7850 | 6332 | 7740 | 43 |
| 18 | 5779 | 8161 | 5920 | 8059 | 6060 | 7955 | 6198 | 7848 | 6334 | 7738 | 42 |
| 19 | 5781 | 8160 | 5922 | 8058 | 6062 | 7953 | 6200 | 7846 | 6336 | 7737 | 41 |
| 20 | 5783 | 8158 | 5925 | 8056 | 6065 | 7951 | 6202 | 7844 | 6338 | 7735 | 40 |
| 21 | 5786 | 8156 | 5927 | 8054 | 6067 | 7950 | 6205 | 7842 | 6341 | 7733 | 39 |
| 22 | 5788 | 8155 | 5930 | 8052 | 6069 | 7948 | 6207 | 7841 | 6343 | 7731 | 38 |
| 23 | 5790 | 8153 | 5932 | 8051 | 6071 | 7946 | 6209 | 7839 | 6345 | 7729 | 37 |
| 24 | 5793 | 8151 | 5934 | 8049 | 6074 | 7944 | 6211 | 7837 | 6347 | 7727 | 36 |
| 25 | 5795 | 8150 | 5937 | 8047 | 6076 | 7942 | 6214 | 7835 | 6350 | 7725 | 35 |
| 26 | 5798 | 8148 | 5939 | 8045 | 6078 | 7941 | 6216 | 7833 | 6352 | 7724 | 34 |
| 27 | 5800 | 8146 | 5941 | 8044 | 6081 | 7939 | 6218 | 7832 | 6354 | 7722 | 33 |
| 28 | 5802 | 8145 | 5944 | 8042 | 6083 | 7937 | 6221 | 7830 | 6356 | 7720 | 32 |
| 29 | 5805 | 8143 | 5946 | 8040 | 6085 | 7935 | 6223 | 7828 | 6359 | 7718 | 31 |
| 30 | 5807 | 8141 | 5948 | 8039 | 6088 | 7934 | 6225 | 7826 | 6361 | 7716 | 30 |
| 31 | 5809 | 8139 | 5951 | 8037 | 6090 | 7932 | 6227 | 7824 | 6363 | 7714 | 29 |
| 32 | 5812 | 8138 | 5953 | 8035 | 6092 | 7930 | 6230 | 7822 | 6365 | 7713 | 28 |
| 33 | 5814 | 8136 | 5955 | 8033 | 6095 | 7928 | 6232 | 7821 | 6368 | 7711 | 27 |
| 34 | 5816 | 8134 | 5958 | 8032 | 6097 | 7926 | 6234 | 7819 | 6370 | 7709 | 26 |
| 35 | 5819 | 8133 | 5960 | 8030 | 6099 | 7925 | 6237 | 7817 | 6372 | 7707 | 25 |
| 36 | 5821 | 8131 | 5962 | 8028 | 6101 | 7923 | 6239 | 7815 | 6374 | 7705 | 24 |
| 37 | 5824 | 8129 | 5965 | 8026 | 6104 | 7921 | 6241 | 7813 | 6376 | 7703 | 23 |
| 38 | 5826 | 8128 | 5967 | 8025 | 6106 | 7919 | 6243 | 7812 | 6379 | 7701 | 22 |
| 39 | 5828 | 8126 | 5969 | 8023 | 6108 | 7918 | 6246 | 7810 | 6381 | 7700 | 21 |
| 40 | 5831 | 8124 | 5972 | 8021 | 6111 | 7916 | 6248 | 7808 | 6383 | 7698 | 20 |
| 41 | 5833 | 8123 | 5974 | 8020 | 6113 | 7914 | 6250 | 7806 | 6385 | 7696 | 19 |
| 42 | 5835 | 8121 | 5976 | 8018 | 6115 | 7912 | 6252 | 7804 | 6388 | 7694 | 18 |
| 43 | 5838 | 8119 | 5979 | 8016 | 6118 | 7910 | 6255 | 7802 | 6390 | 7692 | 17 |
| 44 | 5840 | 8117 | 5981 | 8014 | 6120 | 7909 | 6257 | 7801 | 6392 | 7690 | 16 |
| 45 | 5842 | 8116 | 5983 | 8013 | 6122 | 7907 | 6259 | 7799 | 6394 | 7688 | 15 |
| 46 | 5845 | 8114 | 5986 | 8011 | 6124 | 7905 | 6262 | 7797 | 6397 | 7687 | 14 |
| 47 | 5847 | 8112 | 5988 | 8009 | 6127 | 7903 | 6264 | 7795 | 6399 | 7685 | 13 |
| 48 | 5850 | 8111 | 5990 | 8007 | 6129 | 7902 | 6266 | 7793 | 6401 | 7683 | 12 |
| 49 | 5852 | 8109 | 5993 | 8006 | 6131 | 7900 | 6268 | 7792 | 6403 | 7681 | 11 |
| 50 | 5854 | 8107 | 5995 | 8004 | 6134 | 7898 | 6271 | 7790 | 6406 | 7679 | 10 |
| 51 | 5857 | 8106 | 5997 | 8002 | 6136 | 7896 | 6273 | 7788 | 6408 | 7677 | 9 |
| 52 | 5859 | 8104 | 6000 | 8000 | 6138 | 7894 | 6275 | 7786 | 6410 | 7675 | 8 |
| 53 | 5861 | 8102 | 6002 | 7999 | 6141 | 7893 | 6277 | 7784 | 6412 | 7674 | 7 |
| 54 | 5864 | 8100 | 6004 | 7997 | 6143 | 7891 | 6280 | 7782 | 6414 | 7672 | 6 |
| 55 | 5866 | 8099 | 6007 | 7995 | 6145 | 7889 | 6282 | 7781 | 6417 | 7670 | 5 |
| 56 | 5868 | 8097 | 6009 | 7993 | 6147 | 7887 | 6284 | 7779 | 6419 | 7668 | 4 |
| 57 | 5871 | 8095 | 6011 | 7992 | 6150 | 7885 | 6286 | 7777 | 6421 | 7666 | 3 |
| 58 | 5873 | 8094 | 6014 | 7990 | 6152 | 7884 | 6289 | 7775 | 6423 | 7664 | 2 |
| 59 | 5875 | 8092 | 6016 | 7988 | 6154 | 7882 | 6291 | 7773 | 6426 | 7662 | 1 |
| 60 | 5878 | 8090 | 6018 | 7986 | 6157 | 7880 | 6293 | 7771 | 6428 | 7660 | 0 |
| | cos | sin | cos | sin | cos | sin | cos | sin | cos | sin | |
| ′ | 54° | | 53° | | 52° | | 51° | | 50° | | ′ |

| ′ | 40° | | 41° | | 42° | | 43° | | 44° | | ′ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | sin | cos | sin | cos | sin | cos | sin | cos | sin | cos | |
| 0 | 6428 | 7660 | 6561 | 7547 | 6691 | 7431 | 6820 | 7314 | 6947 | 7193 | 60 |
| 1 | 6430 | 7659 | 6563 | 7545 | 6693 | 7430 | 6822 | 7312 | 6949 | 7191 | 59 |
| 2 | 6432 | 7657 | 6565 | 7543 | 6696 | 7428 | 6824 | 7310 | 6951 | 7189 | 58 |
| 3 | 6435 | 7655 | 6567 | 7541 | 6698 | 7426 | 6826 | 7308 | 6953 | 7187 | 57 |
| 4 | 6437 | 7653 | 6569 | 7539 | 6700 | 7424 | 6828 | 7306 | 6955 | 7185 | 56 |
| 5 | 6439 | 7651 | 6572 | 7538 | 6702 | 7422 | 6831 | 7304 | 6957 | 7183 | 55 |
| 6 | 6441 | 7649 | 6574 | 7536 | 6704 | 7420 | 6833 | 7302 | 6959 | 7181 | 54 |
| 7 | 6443 | 7647 | 6576 | 7534 | 6706 | 7418 | 6835 | 7300 | 6961 | 7179 | 53 |
| 8 | 6446 | 7645 | 6578 | 7532 | 6709 | 7416 | 6837 | 7298 | 6963 | 7177 | 52 |
| 9 | 6448 | 7644 | 6580 | 7530 | 6711 | 7414 | 6839 | 7296 | 6965 | 7175 | 51 |
| 10 | 6450 | 7642 | 6583 | 7528 | 6713 | 7412 | 6841 | 7294 | 6967 | 7173 | 50 |
| 11 | 6452 | 7640 | 6585 | 7526 | 6715 | 7410 | 6843 | 7292 | 6970 | 7171 | 49 |
| 12 | 6455 | 7638 | 6587 | 7524 | 6717 | 7408 | 6845 | 7290 | 6972 | 7169 | 48 |
| 13 | 6457 | 7636 | 6589 | 7522 | 6719 | 7406 | 6848 | 7288 | 6974 | 7167 | 47 |
| 14 | 6459 | 7634 | 6591 | 7520 | 6722 | 7404 | 6850 | 7286 | 6976 | 7165 | 46 |
| 15 | 6461 | 7632 | 6593 | 7518 | 6724 | 7402 | 6852 | 7284 | 6978 | 7163 | 45 |
| 16 | 6463 | 7630 | 6596 | 7516 | 6726 | 7400 | 6854 | 7282 | 6980 | 7161 | 44 |
| 17 | 6466 | 7629 | 6598 | 7515 | 6728 | 7398 | 6856 | 7280 | 6982 | 7159 | 43 |
| 18 | 6468 | 7627 | 6600 | 7513 | 6730 | 7396 | 6858 | 7278 | 6984 | 7157 | 42 |
| 19 | 6470 | 7625 | 6602 | 7511 | 6732 | 7394 | 6860 | 7276 | 6986 | 7155 | 41 |
| 20 | 6472 | 7623 | 6604 | 7509 | 6734 | 7392 | 6862 | 7274 | 6988 | 7153 | 40 |
| 21 | 6475 | 7621 | 6607 | 7507 | 6737 | 7390 | 6865 | 7272 | 6990 | 7151 | 39 |
| 22 | 6477 | 7619 | 6609 | 7505 | 6739 | 7388 | 6867 | 7270 | 6992 | 7149 | 38 |
| 23 | 6479 | 7617 | 6611 | 7503 | 6741 | 7387 | 6869 | 7268 | 6995 | 7147 | 37 |
| 24 | 6481 | 7615 | 6613 | 7501 | 6743 | 7385 | 6871 | 7266 | 6997 | 7145 | 36 |
| 25 | 6483 | 7613 | 6615 | 7499 | 6745 | 7383 | 6873 | 7264 | 6999 | 7143 | 35 |
| 26 | 6486 | 7612 | 6617 | 7497 | 6747 | 7381 | 6875 | 7262 | 7001 | 7141 | 34 |
| 27 | 6488 | 7610 | 6620 | 7495 | 6749 | 7379 | 6877 | 7260 | 7003 | 7139 | 33 |
| 28 | 6490 | 7608 | 6622 | 7493 | 6752 | 7377 | 6879 | 7258 | 7005 | 7137 | 32 |
| 29 | 6492 | 7606 | 6624 | 7491 | 6754 | 7375 | 6881 | 7256 | 7007 | 7135 | 31 |
| 30 | 6494 | 7604 | 6626 | 7490 | 6756 | 7373 | 6884 | 7254 | 7009 | 7133 | 30 |
| 31 | 6497 | 7602 | 6628 | 7488 | 6758 | 7371 | 6886 | 7252 | 7011 | 7130 | 29 |
| 32 | 6499 | 7600 | 6631 | 7486 | 6760 | 7369 | 6888 | 7250 | 7013 | 7128 | 28 |
| 33 | 6501 | 7598 | 6633 | 7484 | 6762 | 7367 | 6890 | 7248 | 7015 | 7126 | 27 |
| 34 | 6503 | 7596 | 6635 | 7482 | 6764 | 7365 | 6892 | 7246 | 7017 | 7124 | 26 |
| 35 | 6506 | 7595 | 6637 | 7480 | 6767 | 7363 | 6894 | 7244 | 7019 | 7122 | 25 |
| 36 | 6508 | 7593 | 6639 | 7478 | 6769 | 7361 | 6896 | 7242 | 7022 | 7120 | 24 |
| 37 | 6510 | 7591 | 6641 | 7476 | 6771 | 7359 | 6898 | 7240 | 7024 | 7118 | 23 |
| 38 | 6512 | 7589 | 6644 | 7474 | 6773 | 7357 | 6900 | 7238 | 7026 | 7116 | 22 |
| 39 | 6514 | 7587 | 6646 | 7472 | 6775 | 7355 | 6903 | 7236 | 7028 | 7114 | 21 |
| 40 | 6517 | 7585 | 6648 | 7470 | 6777 | 7353 | 6905 | 7234 | 7030 | 7112 | 20 |
| 41 | 6519 | 7583 | 6650 | 7468 | 6779 | 7351 | 6907 | 7232 | 7032 | 7110 | 19 |
| 42 | 6521 | 7581 | 6652 | 7466 | 6782 | 7349 | 6909 | 7230 | 7034 | 7108 | 18 |
| 43 | 6523 | 7579 | 6654 | 7464 | 6784 | 7347 | 6911 | 7228 | 7036 | 7106 | 17 |
| 44 | 6525 | 7578 | 6657 | 7463 | 6786 | 7345 | 6913 | 7226 | 7038 | 7104 | 16 |
| 45 | 6528 | 7576 | 6659 | 7461 | 6788 | 7343 | 6915 | 7224 | 7040 | 7102 | 15 |
| 46 | 6530 | 7574 | 6661 | 7459 | 6790 | 7341 | 6917 | 7222 | 7042 | 7100 | 14 |
| 47 | 6532 | 7572 | 6663 | 7457 | 6792 | 7339 | 6919 | 7220 | 7044 | 7098 | 13 |
| 48 | 6534 | 7570 | 6665 | 7455 | 6794 | 7337 | 6921 | 7218 | 7046 | 7096 | 12 |
| 49 | 6536 | 7568 | 6667 | 7453 | 6797 | 7335 | 6924 | 7216 | 7048 | 7094 | 11 |
| 50 | 6539 | 7566 | 6670 | 7451 | 6799 | 7333 | 6926 | 7214 | 7050 | 7092 | 10 |
| 51 | 6541 | 7564 | 6672 | 7449 | 6801 | 7331 | 6928 | 7212 | 7053 | 7090 | 9 |
| 52 | 6543 | 7562 | 6674 | 7447 | 6803 | 7329 | 6930 | 7210 | 7055 | 7088 | 8 |
| 53 | 6545 | 7560 | 6676 | 7445 | 6805 | 7327 | 6932 | 7208 | 7057 | 7085 | 7 |
| 54 | 6547 | 7559 | 6678 | 7443 | 6807 | 7325 | 6934 | 7206 | 7059 | 7083 | 6 |
| 55 | 6550 | 7557 | 6680 | 7441 | 6809 | 7323 | 6936 | 7203 | 7061 | 7081 | 5 |
| 56 | 6552 | 7555 | 6683 | 7439 | 6811 | 7321 | 6938 | 7201 | 7063 | 7079 | 4 |
| 57 | 6554 | 7553 | 6685 | 7437 | 6814 | 7319 | 6940 | 7199 | 7065 | 7077 | 3 |
| 58 | 6556 | 7551 | 6687 | 7435 | 6816 | 7318 | 6942 | 7197 | 7067 | 7075 | 2 |
| 59 | 6558 | 7549 | 6689 | 7433 | 6818 | 7316 | 6944 | 7195 | 7069 | 7073 | 1 |
| 60 | 6561 | 7547 | 6691 | 7431 | 6820 | 7314 | 6947 | 7193 | 7071 | 7071 | 0 |
| | cos | sin | cos | sin | cos | sin | cos | sin | cos | sin | |
| ′ | 49° | | 48° | | 47° | | 46° | | 45° | | ′ |

| ′ | 0° | | 1° | | 2° | | 3° | | 4° | | ′ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | tan | cot | tan | cot | tan | cot | tan | cot | tan | cot | |
| 0 | 0000 | Infinite | 0175 | 57.2900 | 0349 | 28.6363 | 0524 | 19.0811 | 0699 | 14.3007 | 60 |
| 1 | 0003 | 3437.75 | 0177 | 56.3506 | 0352 | 28.3994 | 0527 | 18.9755 | 0702 | 14.2411 | 59 |
| 2 | 0006 | 1718.87 | 0180 | 55.4415 | 0355 | 28.1664 | 0530 | 18.8711 | 0705 | 14.1821 | 58 |
| 3 | 0009 | 1145.92 | 0183 | 54.5613 | 0358 | 27.9372 | 0533 | 18.7678 | 0708 | 14.1235 | 57 |
| 4 | 0012 | 859.436 | 0186 | 53.7086 | 0361 | 27.7117 | 0536 | 18.6656 | 0711 | 14.0655 | 56 |
| 5 | 0015 | 687.549 | 0189 | 52.8821 | 0364 | 27.4899 | 0539 | 18.5645 | 0714 | 14.0079 | 55 |
| 6 | 0017 | 572.957 | 0192 | 52.0807 | 0367 | 27.2715 | 0542 | 18.4645 | 0717 | 13.9507 | 54 |
| 7 | 0020 | 491.106 | 0195 | 51.3032 | 0370 | 27.0566 | 0544 | 18.3655 | 0720 | 13.8940 | 53 |
| 8 | 0023 | 429.718 | 0198 | 50.5485 | 0373 | 26.8450 | 0547 | 18.2677 | 0723 | 13.8378 | 52 |
| 9 | 0026 | 381.971 | 0201 | 49.8157 | 0375 | 26.6367 | 0550 | 18.1708 | 0726 | 13.7821 | 51 |
| 10 | 0029 | 343.774 | 0204 | 49.1039 | 0378 | 26.4316 | 0553 | 18.0750 | 0729 | 13.7267 | 50 |
| 11 | 0032 | 312.521 | 0207 | 48.4121 | 0381 | 26.2296 | 0556 | 17.9802 | 0731 | 13.6719 | 49 |
| 12 | 0035 | 286.478 | 0209 | 47.7395 | 0384 | 26.0307 | 0559 | 17.8863 | 0734 | 13.6174 | 48 |
| 13 | 0038 | 264.441 | 0212 | 47.0853 | 0387 | 25.8348 | 0562 | 17.7934 | 0737 | 13.5634 | 47 |
| 14 | 0041 | 245.552 | 0215 | 46.4489 | 0390 | 25.6418 | 0565 | 17.7015 | 0740 | 13.5098 | 46 |
| 15 | 0044 | 229.182 | 0218 | 45.8294 | 0393 | 25.4517 | 0568 | 17.6106 | 0743 | 13.4566 | 45 |
| 16 | 0047 | 214.858 | 0221 | 45.2261 | 0396 | 25.2644 | 0571 | 17.5205 | 0746 | 13.4039 | 44 |
| 17 | 0049 | 202.219 | 0224 | 44.6386 | 0399 | 25.0798 | 0574 | 17.4314 | 0749 | 13.3515 | 43 |
| 18 | 0052 | 190.984 | 0227 | 44.0661 | 0402 | 24.8978 | 0577 | 17.3432 | 0752 | 13.2996 | 42 |
| 19 | 0055 | 180.932 | 0230 | 43.5081 | 0405 | 24.7185 | 0580 | 17.2558 | 0755 | 13.2480 | 41 |
| 20 | 0058 | 171.885 | 0233 | 42.9641 | 0407 | 24.5418 | 0582 | 17.1693 | 0758 | 13.1969 | 40 |
| 21 | 0061 | 163.700 | 0236 | 42.4335 | 0410 | 24.3675 | 0585 | 17.0837 | 0761 | 13.1461 | 39 |
| 22 | 0064 | 156.259 | 0239 | 41.9158 | 0413 | 24.1957 | 0588 | 16.9990 | 0764 | 13.0958 | 38 |
| 23 | 0067 | 149.465 | 0241 | 41.4106 | 0416 | 24.0263 | 0591 | 16.9150 | 0767 | 13.0458 | 37 |
| 24 | 0070 | 143.237 | 0244 | 40.9174 | 0419 | 23.8593 | 0594 | 16.8319 | 0769 | 12.9962 | 36 |
| 25 | 0073 | 137.507 | 0247 | 40.4358 | 0422 | 23.6945 | 0597 | 16.7496 | 0772 | 12.9469 | 35 |
| 26 | 0076 | 132.219 | 0250 | 39.9655 | 0425 | 23.5321 | 0600 | 16.6681 | 0775 | 12.8981 | 34 |
| 27 | 0079 | 127.321 | 0253 | 39.5059 | 0428 | 23.3718 | 0603 | 16.5874 | 0778 | 12.8496 | 33 |
| 28 | 0081 | 122.774 | 0256 | 39.0568 | 0431 | 23.2137 | 0606 | 16.5075 | 0781 | 12.8014 | 32 |
| 29 | 0084 | 118.540 | 0259 | 38.6177 | 0434 | 23.0577 | 0609 | 16.4283 | 0784 | 12.7536 | 31 |
| 30 | 0087 | 114.589 | 0262 | 38.1885 | 0437 | 22.9038 | 0612 | 16.3499 | 0787 | 12.7062 | 30 |
| 31 | 0090 | 110.892 | 0265 | 37.7686 | 0440 | 22.7519 | 0615 | 16.2722 | 0790 | 12.6591 | 29 |
| 32 | 0093 | 107.426 | 0268 | 37.3579 | 0442 | 22.6020 | 0617 | 16.1952 | 0793 | 12.6124 | 28 |
| 33 | 0096 | 104.171 | 0271 | 36.9560 | 0445 | 22.4541 | 0620 | 16.1190 | 0796 | 12.5660 | 27 |
| 34 | 0099 | 101.107 | 0274 | 36.5627 | 0448 | 22.3081 | 0623 | 16.0435 | 0799 | 12.5199 | 26 |
| 35 | 0102 | 98.2179 | 0276 | 36.1776 | 0451 | 22.1640 | 0626 | 15.9687 | 0802 | 12.4742 | 25 |
| 36 | 0105 | 95.4895 | 0279 | 35.8006 | 0454 | 22.0217 | 0629 | 15.8945 | 0805 | 12.4288 | 24 |
| 37 | 0108 | 92.9085 | 0282 | 35.4313 | 0457 | 21.8813 | 0632 | 15.8211 | 0808 | 12.3838 | 23 |
| 38 | 0111 | 90.4633 | 0285 | 35.0695 | 0460 | 21.7426 | 0635 | 15.7483 | 0810 | 12.3390 | 22 |
| 39 | 0113 | 88.1436 | 0288 | 34.7151 | 0463 | 21.6056 | 0638 | 15.6762 | 0813 | 12.2946 | 21 |
| 40 | 0116 | 85.9398 | 0291 | 34.3678 | 0466 | 21.4704 | 0641 | 15.6048 | 0816 | 12.2505 | 20 |
| 41 | 0119 | 83.8435 | 0294 | 34.0273 | 0469 | 21.3369 | 0644 | 15.5340 | 0819 | 12.2067 | 19 |
| 42 | 0122 | 81.8470 | 0297 | 33.6935 | 0472 | 21.2049 | 0647 | 15.4638 | 0822 | 12.1632 | 18 |
| 43 | 0125 | 79.9434 | 0300 | 33.3662 | 0475 | 21.0747 | 0650 | 15.3943 | 0825 | 12.1201 | 17 |
| 44 | 0128 | 78.1263 | 0303 | 33.0452 | 0477 | 20.9460 | 0653 | 15.3254 | 0828 | 12.0772 | 16 |
| 45 | 0131 | 76.3900 | 0306 | 32.7303 | 0480 | 20.8188 | 0655 | 15.2571 | 0831 | 12.0346 | 15 |
| 46 | 0134 | 74.7292 | 0308 | 32.4213 | 0483 | 20.6932 | 0658 | 15.1893 | 0834 | 11.9923 | 14 |
| 47 | 0137 | 73.1390 | 0311 | 32.1181 | 0486 | 20.5691 | 0661 | 15.1222 | 0837 | 11.9504 | 13 |
| 48 | 0140 | 71.6151 | 0314 | 31.8205 | 0489 | 20.4465 | 0664 | 15.0557 | 0840 | 11.9087 | 12 |
| 49 | 0143 | 70.1533 | 0317 | 31.5284 | 0492 | 20.3253 | 0667 | 14.9898 | 0843 | 11.8673 | 11 |
| 50 | 0146 | 68.7501 | 0320 | 31.2416 | 0495 | 20.2056 | 0670 | 14.9244 | 0846 | 11.8262 | 10 |
| 51 | 0148 | 67.4019 | 0323 | 30.9599 | 0498 | 20.0872 | 0673 | 14.8596 | 0849 | 11.7853 | 9 |
| 52 | 0151 | 66.1055 | 0326 | 30.6833 | 0501 | 19.9702 | 0676 | 14.7954 | 0851 | 11.7448 | 8 |
| 53 | 0154 | 64.8580 | 0329 | 30.4116 | 0504 | 19.8546 | 0679 | 14.7317 | 0854 | 11.7045 | 7 |
| 54 | 0157 | 63.6567 | 0332 | 30.1446 | 0507 | 19.7403 | 0682 | 14.6685 | 0857 | 11.6645 | 6 |
| 55 | 0160 | 62.4992 | 0335 | 29.8823 | 0509 | 19.6273 | 0685 | 14.6059 | 0860 | 11.6248 | 5 |
| 56 | 0163 | 61.3829 | 0338 | 29.6245 | 0512 | 19.5156 | 0688 | 14.5438 | 0863 | 11.5853 | 4 |
| 57 | 0166 | 60.3058 | 0340 | 29.3711 | 0515 | 19.4051 | 0690 | 14.4823 | 0866 | 11.5461 | 3 |
| 58 | 0169 | 59.2659 | 0343 | 29.1220 | 0518 | 19.2959 | 0693 | 14.4212 | 0869 | 11.5072 | 2 |
| 59 | 0172 | 58.2612 | 0346 | 28.8771 | 0521 | 19.1879 | 0696 | 14.3607 | 0872 | 11.4685 | 1 |
| 60 | 0175 | 57.2900 | 0349 | 28.6363 | 0524 | 19.0811 | 0699 | 14.3007 | 0875 | 11.4301 | 0 |
| | cot | tan | cot | tan | cot | tan | cot | tan | cot | tan | |
| ′ | 89° | | 88° | | 87° | | 86° | | 85° | | ′ |

| ′ | 5° | | 6° | | 7° | | 8° | | 9° | | ′ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | tan | cot | tan | cot | tan | cot | tan | cot | tan | cot | |
| **0** | 0875 | 11.4301 | 1051 | 9.5144 | 1228 | 8.1443 | 1405 | 7.1154 | 1584 | 6.3138 | **60** |
| 1 | 0878 | 11.3919 | 1054 | 9.4878 | 1231 | 8.1248 | 1408 | 7.1004 | 1587 | 6.3019 | 59 |
| 2 | 0881 | 11.3540 | 1057 | 9.4614 | 1234 | 8.1054 | 1411 | 7.0855 | 1590 | 6.2901 | 58 |
| 3 | 0884 | 11.3163 | 1060 | 9.4352 | 1237 | 8.0860 | 1414 | 7.0706 | 1593 | 6.2783 | 57 |
| 4 | 0887 | 11.2789 | 1063 | 9.4090 | 1240 | 8.0667 | 1417 | 7.0558 | 1596 | 6.2666 | 56 |
| **5** | 0890 | 11.2417 | 1066 | 9.3831 | 1243 | 8.0476 | 1420 | 7.0410 | 1599 | 6.2549 | **55** |
| 6 | 0892 | 11.2048 | 1069 | 9.3572 | 1246 | 8.0285 | 1423 | 7.0264 | 1602 | 6.2432 | 54 |
| 7 | 0895 | 11.1681 | 1072 | 9.3315 | 1249 | 8.0095 | 1426 | 7.0117 | 1605 | 6.2316 | 53 |
| 8 | 0898 | 11.1316 | 1075 | 9.3060 | 1251 | 7.9906 | 1429 | 6.9972 | 1608 | 6.2200 | 52 |
| 9 | 0901 | 11.0954 | 1078 | 9.2806 | 1254 | 7.9718 | 1432 | 6.9827 | 1611 | 6.2085 | 51 |
| **10** | 0904 | 11.0594 | 1080 | 9.2553 | 1257 | 7.9530 | 1435 | 6.9682 | 1614 | 6.1970 | **50** |
| 11 | 0907 | 11.0237 | 1083 | 9.2302 | 1260 | 7.9344 | 1438 | 6.9538 | 1617 | 6.1856 | 49 |
| 12 | 0910 | 10.9882 | 1086 | 9.2052 | 1263 | 7.9158 | 1441 | 6.9395 | 1620 | 6.1742 | 48 |
| 13 | 0913 | 10.9529 | 1089 | 9.1803 | 1266 | 7.8973 | 1444 | 6.9252 | 1623 | 6.1628 | 47 |
| 14 | 0916 | 10.9178 | 1092 | 9.1555 | 1269 | 7.8789 | 1447 | 6.9110 | 1626 | 6.1515 | 46 |
| **15** | 0919 | 10.8829 | 1095 | 9.1309 | 1272 | 7.8606 | 1450 | 6.8969 | 1629 | 6.1402 | **45** |
| 16 | 0922 | 10.8483 | 1098 | 9.1065 | 1275 | 7.8424 | 1453 | 6.8828 | 1632 | 6.1290 | 44 |
| 17 | 0925 | 10.8139 | 1101 | 9.0821 | 1278 | 7.8243 | 1456 | 6.8687 | 1635 | 6.1178 | 43 |
| 18 | 0928 | 10.7797 | 1104 | 9.0579 | 1281 | 7.8062 | 1459 | 6.8548 | 1638 | 6.1066 | 42 |
| 19 | 0931 | 10.7457 | 1107 | 9.0338 | 1284 | 7.7883 | 1462 | 6.8408 | 1641 | 6.0955 | 41 |
| **20** | 0934 | 10.7119 | 1110 | 9.0098 | 1287 | 7.7704 | 1465 | 6.8269 | 1644 | 6.0844 | **40** |
| 21 | 0936 | 10.6783 | 1113 | 8.9860 | 1290 | 7.7525 | 1468 | 6.8131 | 1647 | 6.0734 | 39 |
| 22 | 0939 | 10.6450 | 1116 | 8.9623 | 1293 | 7.7348 | 1471 | 6.7994 | 1650 | 6.0624 | 38 |
| 23 | 0942 | 10.6118 | 1119 | 8.9387 | 1296 | 7.7171 | 1474 | 6.7856 | 1653 | 6.0514 | 37 |
| 24 | 0945 | 10.5789 | 1122 | 8.9152 | 1299 | 7.6996 | 1477 | 6.7720 | 1655 | 6.0405 | 36 |
| **25** | 0948 | 10.5462 | 1125 | 8.8919 | 1302 | 7.6821 | 1480 | 6.7584 | 1658 | 6.0296 | **35** |
| 26 | 0951 | 10.5136 | 1128 | 8.8686 | 1305 | 7.6647 | 1483 | 6.7448 | 1661 | 6.0188 | 34 |
| 27 | 0954 | 10.4813 | 1131 | 8.8455 | 1308 | 7.6473 | 1486 | 6.7313 | 1664 | 6.0080 | 33 |
| 28 | 0957 | 10.4491 | 1134 | 8.8225 | 1311 | 7.6301 | 1489 | 6.7179 | 1667 | 5.9972 | 32 |
| 29 | 0960 | 10.4172 | 1136 | 8.7996 | 1314 | 7.6129 | 1492 | 6.7045 | 1670 | 5.9865 | 31 |
| **30** | 0963 | 10.3854 | 1139 | 8.7769 | 1317 | 7.5958 | 1495 | 6.6912 | 1673 | 5.9758 | **30** |
| 31 | 0966 | 10.3538 | 1142 | 8.7542 | 1319 | 7.5787 | 1497 | 6.6779 | 1676 | 5.9651 | 29 |
| 32 | 0969 | 10.3224 | 1145 | 8.7317 | 1322 | 7.5618 | 1500 | 6.6646 | 1679 | 5.9545 | 28 |
| 33 | 0972 | 10.2913 | 1148 | 8.7093 | 1325 | 7.5449 | 1503 | 6.6514 | 1682 | 5.9439 | 27 |
| 34 | 0975 | 10.2602 | 1151 | 8.6870 | 1328 | 7.5281 | 1506 | 6.6383 | 1685 | 5.9333 | 26 |
| **35** | 0978 | 10.2294 | 1154 | 8.6648 | 1331 | 7.5113 | 1509 | 6.6252 | 1688 | 5.9228 | **25** |
| 36 | 0981 | 10.1988 | 1157 | 8.6427 | 1334 | 7.4947 | 1512 | 6.6122 | 1691 | 5.9124 | 24 |
| 37 | 0983 | 10.1683 | 1160 | 8.6208 | 1337 | 7.4781 | 1515 | 6.5992 | 1694 | 5.9019 | 23 |
| 38 | 0986 | 10.1381 | 1163 | 8.5989 | 1340 | 7.4615 | 1518 | 6.5863 | 1697 | 5.8915 | 22 |
| 39 | 0989 | 10.1080 | 1166 | 8.5772 | 1343 | 7.4451 | 1521 | 6.5734 | 1700 | 5.8811 | 21 |
| **40** | 0992 | 10.0780 | 1169 | 8.5555 | 1346 | 7.4287 | 1524 | 6.5606 | 1703 | 5.8708 | **20** |
| 41 | 0995 | 10.0483 | 1172 | 8.5340 | 1349 | 7.4124 | 1527 | 6.5478 | 1706 | 5.8605 | 19 |
| 42 | 0998 | 10.0187 | 1175 | 8.5126 | 1352 | 7.3962 | 1530 | 6.5350 | 1709 | 5.8502 | 18 |
| 43 | 1001 | 9.9893 | 1178 | 8.4913 | 1355 | 7.3800 | 1533 | 6.5223 | 1712 | 5.8400 | 17 |
| 44 | 1004 | 9.9601 | 1181 | 8.4701 | 1358 | 7.3639 | 1536 | 6.5097 | 1715 | 5.8298 | 16 |
| **45** | 1007 | 9.9310 | 1184 | 8.4490 | 1361 | 7.3479 | 1539 | 6.4971 | 1718 | 5.8197 | **15** |
| 46 | 1010 | 9.9021 | 1187 | 8.4280 | 1364 | 7.3319 | 1542 | 6.4846 | 1721 | 5.8095 | 14 |
| 47 | 1013 | 9.8734 | 1189 | 8.4071 | 1367 | 7.3160 | 1545 | 6.4721 | 1724 | 5.7994 | 13 |
| 48 | 1016 | 9.8448 | 1192 | 8.3863 | 1370 | 7.3002 | 1548 | 6.4596 | 1727 | 5.7894 | 12 |
| 49 | 1019 | 9.8164 | 1195 | 8.3656 | 1373 | 7.2844 | 1551 | 6.4472 | 1730 | 5.7794 | 12 |
| **50** | 1022 | 9.7882 | 1198 | 8.3450 | 1376 | 7.2687 | 1554 | 6.4348 | 1733 | 5.7694 | **10** |
| 51 | 1025 | 9.7601 | 1201 | 8.3245 | 1379 | 7.2531 | 1557 | 6.4225 | 1736 | 5.7594 | 9 |
| 52 | 1028 | 9.7322 | 1204 | 8.3041 | 1382 | 7.2375 | 1560 | 6.4103 | 1739 | 5.7495 | 8 |
| 53 | 1030 | 9.7044 | 1207 | 8.2838 | 1385 | 7.2220 | 1563 | 6.3980 | 1742 | 5.7396 | 7 |
| 54 | 1033 | 9.6768 | 1210 | 8.2636 | 1388 | 7.2066 | 1566 | 6.3859 | 1745 | 5.7297 | 6 |
| **55** | 1036 | 9.6499 | 1213 | 8.2434 | 1391 | 7.1912 | 1569 | 6.3737 | 1748 | 5.7199 | **5** |
| 56 | 1039 | 9.6220 | 1216 | 8.2234 | 1394 | 7.1759 | 1572 | 6.3617 | 1751 | 5.7101 | 4 |
| 57 | 1042 | 9.5949 | 1219 | 8.2035 | 1397 | 7.1607 | 1575 | 6.3496 | 1754 | 5.7004 | 3 |
| 58 | 1045 | 9.5679 | 1222 | 8.1837 | 1399 | 7.1455 | 1578 | 6.3376 | 1757 | 5.6906 | 2 |
| 59 | 1048 | 9.5411 | 1225 | 8.1640 | 1402 | 7.1304 | 1581 | 6.3257 | 1760 | 5.6809 | 1 |
| **60** | 1051 | 9.5144 | 1228 | 8.1443 | 1405 | 7.1154 | 1584 | 6.3138 | 1763 | 5.6713 | **0** |
| | cot | tan | cot | tan | cot | tan | cot | tan | cot | tan | |
| ′ | 84° | | 83° | | 82° | | 81° | | 80° | | ′ |

| ′ | 10° | | 11° | | 12° | | 13° | | 14° | | ′ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | tan | cot | tan | cot | tan | cot | tan | cot | tan | cot | |
| **0** | 1763 | 5.6713 | 1944 | 5.1446 | 2126 | 4.7046 | 2309 | 4.3315 | 2493 | 4.0108 | **60** |
| 1 | 1766 | 5.6617 | 1947 | 5.1366 | 2129 | 4.6979 | 2312 | 4.3257 | 2496 | 4.0058 | 59 |
| 2 | 1769 | 5.6521 | 1950 | 5.1286 | 2132 | 4.6912 | 2315 | 4.3200 | 2499 | 4.0009 | 58 |
| 3 | 1772 | 5.6425 | 1953 | 5.1207 | 2135 | 4.6845 | 2318 | 4.3143 | 2503 | 3.9959 | 57 |
| 4 | 1775 | 5.6330 | 1956 | 5.1128 | 2138 | 4.6779 | 2321 | 4.3086 | 2506 | 3.9910 | 56 |
| **5** | 1778 | 5.6234 | 1959 | 5.1049 | 2141 | 4.6712 | 2324 | 4.3029 | 2509 | 3.9861 | **55** |
| 6 | 1781 | 5.6140 | 1962 | 5.0970 | 2144 | 4.6646 | 2327 | 4.2972 | 2512 | 3.9812 | 54 |
| 7 | 1784 | 5.6045 | 1965 | 5.0892 | 2147 | 4.6580 | 2330 | 4.2916 | 2515 | 3.9763 | 53 |
| 8 | 1787 | 5.5951 | 1968 | 5.0814 | 2150 | 4.6514 | 2333 | 4.2859 | 2518 | 3.9714 | 52 |
| 9 | 1790 | 5.5857 | 1971 | 5.0736 | 2153 | 4.6448 | 2336 | 4.2803 | 2521 | 3.9665 | 51 |
| **10** | 1793 | 5.5764 | 1974 | 5.0658 | 2156 | 4.6382 | 2339 | 4.2747 | 2524 | 3.9617 | **50** |
| 11 | 1796 | 5.5671 | 1977 | 5.0581 | 2159 | 4.6317 | 2342 | 4.2691 | 2527 | 3.9568 | 49 |
| 12 | 1799 | 5.5578 | 1980 | 5.0504 | 2162 | 4.6252 | 2345 | 4.2635 | 2530 | 3.9520 | 48 |
| 13 | 1802 | 5.5485 | 1983 | 5.0427 | 2165 | 4.6187 | 2349 | 4.2580 | 2533 | 3.9471 | 47 |
| 14 | 1805 | 5.5393 | 1986 | 5.0350 | 2168 | 4.6122 | 2352 | 4.2524 | 2537 | 3.9423 | 46 |
| **15** | 1808 | 5.5301 | 1989 | 5.0273 | 2171 | 4.6057 | 2355 | 4.2468 | 2540 | 3.9375 | **45** |
| 16 | 1811 | 5.5209 | 1992 | 5.0197 | 2174 | 4.5993 | 2358 | 4.2413 | 2543 | 3.9327 | 44 |
| 17 | 1814 | 5.5118 | 1995 | 5.0121 | 2177 | 4.5928 | 2361 | 4.2358 | 2546 | 3.9279 | 43 |
| 18 | 1817 | 5.5026 | 1998 | 5.0045 | 2180 | 4.5864 | 2364 | 4.2303 | 2549 | 3.9232 | 42 |
| 19 | 1820 | 5.4936 | 2001 | 4.9969 | 2183 | 4.5800 | 2367 | 4.2248 | 2552 | 3.9184 | 41 |
| **20** | 1823 | 5.4845 | 2004 | 4.9894 | 2186 | 4.5736 | 2370 | 4.2193 | 2555 | 3.9136 | **40** |
| 21 | 1826 | 5.4755 | 2007 | 4.9819 | 2189 | 4.5673 | 2373 | 4.2139 | 2558 | 3.9089 | 39 |
| 22 | 1829 | 5.4665 | 2010 | 4.9744 | 2193 | 4.5609 | 2376 | 4.2084 | 2561 | 3.9042 | 38 |
| 23 | 1832 | 5.4575 | 2013 | 4.9669 | 2196 | 4.5546 | 2379 | 4.2030 | 2564 | 3.8995 | 37 |
| 24 | 1835 | 5.4486 | 2016 | 4.9594 | 2199 | 4.5483 | 2382 | 4.1976 | 2568 | 3.8947 | 36 |
| **25** | 1838 | 5.4397 | 2019 | 4.9520 | 2202 | 4.5420 | 2385 | 4.1922 | 2571 | 3.8900 | **35** |
| 26 | 1841 | 5.4308 | 2022 | 4.9446 | 2205 | 4.5357 | 2388 | 4.1868 | 2574 | 3.8854 | 34 |
| 27 | 1844 | 5.4219 | 2025 | 4.9372 | 2208 | 4.5294 | 2392 | 4.1814 | 2577 | 3.8807 | 33 |
| 28 | 1847 | 5.4131 | 2028 | 4.9298 | 2211 | 4.5232 | 2395 | 4.1760 | 2580 | 3.8760 | 32 |
| 29 | 1850 | 5.4043 | 2031 | 4.9225 | 2214 | 4.5169 | 2398 | 4.1706 | 2583 | 3.8714 | 31 |
| **30** | 1853 | 5.3955 | 2035 | 4.9152 | 2217 | 4.5107 | 2401 | 4.1653 | 2586 | 3.8667 | **30** |
| 31 | 1856 | 5.3868 | 2038 | 4.9078 | 2220 | 4.5045 | 2404 | 4.1600 | 2589 | 3.8621 | 29 |
| 32 | 1859 | 5.3781 | 2041 | 4.9006 | 2223 | 4.4983 | 2407 | 4.1547 | 2592 | 3.8575 | 28 |
| 33 | 1862 | 5.3694 | 2044 | 4.8933 | 2226 | 4.4922 | 2410 | 4.1493 | 2595 | 3.8528 | 27 |
| 34 | 1865 | 5.3607 | 2047 | 4.8860 | 2229 | 4.4860 | 2413 | 4.1441 | 2599 | 3.8482 | 26 |
| **35** | 1868 | 5.3521 | 2050 | 4.8788 | 2232 | 4.4799 | 2416 | 4.1388 | 2602 | 3.8436 | **25** |
| 36 | 1871 | 5.3435 | 2053 | 4.8716 | 2235 | 4.4737 | 2419 | 4.1335 | 2605 | 3.8391 | 24 |
| 37 | 1874 | 5.3349 | 2056 | 4.8644 | 2238 | 4.4676 | 2422 | 4.1282 | 2608 | 3.8345 | 23 |
| 38 | 1877 | 5.3263 | 2059 | 4.8573 | 2241 | 4.4615 | 2425 | 4.1230 | 2611 | 3.8299 | 22 |
| 39 | 1880 | 5.3178 | 2062 | 4.8501 | 2244 | 4.4555 | 2428 | 4.1178 | 2614 | 3.8254 | 21 |
| **40** | 1883 | 5.3093 | 2065 | 4.8430 | 2247 | 4.4494 | 2432 | 4.1126 | 2617 | 3.8208 | **20** |
| 41 | 1887 | 5.3008 | 2068 | 4.8359 | 2251 | 4.4434 | 2435 | 4.1074 | 2620 | 3.8163 | 19 |
| 42 | 1890 | 5.2924 | 2071 | 4.8288 | 2254 | 4.4374 | 2438 | 4.1022 | 2623 | 3.8118 | 18 |
| 43 | 1893 | 5.2839 | 2074 | 4.8218 | 2257 | 4.4313 | 2441 | 4.0970 | 2627 | 3.8073 | 17 |
| 44 | 1896 | 5.2755 | 2077 | 4.8147 | 2260 | 4.4253 | 2444 | 4.0918 | 2630 | 3.8028 | 16 |
| **45** | 1899 | 5.2672 | 2080 | 4.8077 | 2263 | 4.4194 | 2447 | 4.0867 | 2633 | 3.7983 | **15** |
| 46 | 1902 | 5.2588 | 2083 | 4.8007 | 2266 | 4.4134 | 2450 | 4.0815 | 2636 | 3.7938 | 14 |
| 47 | 1905 | 5.2505 | 2086 | 4.7937 | 2269 | 4.4075 | 2453 | 4.0764 | 2639 | 3.7893 | 13 |
| 48 | 1908 | 5.2422 | 2089 | 4.7867 | 2272 | 4.4015 | 2456 | 4.0713 | 2642 | 3.7848 | 12 |
| 49 | 1911 | 5.2339 | 2092 | 4.7798 | 2275 | 4.3956 | 2459 | 4.0662 | 2645 | 3.7804 | 11 |
| **50** | 1914 | 5.2257 | 2095 | 4.7729 | 2278 | 4.3897 | 2462 | 4.0611 | 2648 | 3.7760 | **10** |
| 51 | 1917 | 5.2174 | 2098 | 4.7659 | 2281 | 4.3838 | 2465 | 4.0560 | 2651 | 3.7715 | 9 |
| 52 | 1920 | 5.2092 | 2101 | 4.7591 | 2284 | 4.3779 | 2469 | 4.0509 | 2655 | 3.7671 | 8 |
| 53 | 1923 | 5.2011 | 2104 | 4.7522 | 2287 | 4.3721 | 2472 | 4.0459 | 2658 | 3.7627 | 7 |
| 54 | 1926 | 5.1929 | 2107 | 4.7453 | 2290 | 4.3662 | 2475 | 4.0408 | 2661 | 3.7583 | 6 |
| **55** | 1929 | 5.1848 | 2110 | 4.7385 | 2293 | 4.3604 | 2478 | 4.0358 | 2664 | 3.7539 | **5** |
| 56 | 1932 | 5.1767 | 2113 | 4.7317 | 2296 | 4.3546 | 2481 | 4.0308 | 2667 | 3.7495 | 4 |
| 57 | 1935 | 5.1686 | 2116 | 4.7249 | 2299 | 4.3488 | 2484 | 4.0257 | 2670 | 3.7451 | 3 |
| 58 | 1938 | 5.1606 | 2119 | 4.7181 | 2303 | 4.3430 | 2487 | 4.0207 | 2673 | 3.7408 | 2 |
| 59 | 1941 | 5.1526 | 2123 | 4.7114 | 2306 | 4.3372 | 2490 | 4.0158 | 2676 | 3.7364 | 1 |
| **60** | 1944 | 5.1446 | 2126 | 4.7046 | 2309 | 4.3315 | 2493 | 4.0108 | 2679 | 3.7321 | **0** |
| | cot | tan | cot | tan | cot | tan | cot | tan | cot | tan | |
| ′ | 79° | | 78° | | 77° | | 76° | | 75° | | ′ |

| ' | 15° | | 16° | | 17° | | 18° | | 19° | | ' |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | tan | cot | tan | cot | tan | cot | tan | cot | tan | cot | |
| 0 | 2679 | 3.7321 | 2867 | 3.4874 | 3057 | 3.2709 | 3249 | 3.0777 | 3443 | 2.9042 | 60 |
| 1 | 2683 | 3.7277 | 2871 | 3.4836 | 3060 | 3.2675 | 3252 | 3.0746 | 3447 | 2.9015 | 59 |
| 2 | 2686 | 3.7234 | 2874 | 3.4798 | 3064 | 3.2641 | 3256 | 3.0716 | 3450 | 2.8987 | 58 |
| 3 | 2689 | 3.7191 | 2877 | 3.4760 | 3067 | 3.2607 | 3259 | 3.0686 | 3453 | 2.8960 | 57 |
| 4 | 2692 | 3.7148 | 2880 | 3.4722 | 3070 | 3.2573 | 3262 | 3.0655 | 3456 | 2.8933 | 56 |
| 5 | 2695 | 3.7105 | 2883 | 3.4684 | 3073 | 3.2539 | 3265 | 3.0625 | 3460 | 2.8905 | 55 |
| 6 | 2698 | 3.7062 | 2886 | 3.4646 | 3076 | 3.2506 | 3269 | 3.0595 | 3463 | 2.8878 | 54 |
| 7 | 2701 | 3.7019 | 2890 | 3.4608 | 3080 | 3.2472 | 3272 | 3.0565 | 3466 | 2.8851 | 53 |
| 8 | 2704 | 3.6976 | 2893 | 3.4570 | 3083 | 3.2438 | 3275 | 3.0535 | 3469 | 2.8824 | 52 |
| 9 | 2703 | 3.6933 | 2896 | 3.4533 | 3086 | 3.2405 | 3278 | 3.0505 | 3473 | 2.8797 | 51 |
| 10 | 2711 | 3.6891 | 2899 | 3.4495 | 3089 | 3.2371 | 3281 | 3.0475 | 3476 | 2.8770 | 50 |
| 11 | 2714 | 3.6848 | 2902 | 3.4458 | 3092 | 3.2338 | 3285 | 3.0445 | 3479 | 2.8743 | 49 |
| 12 | 2717 | 3.6806 | 2905 | 3.4420 | 3096 | 3.2305 | 3288 | 3.0415 | 3482 | 2.8716 | 48 |
| 13 | 2720 | 3.6764 | 2908 | 3.4383 | 3099 | 3.2272 | 3291 | 3.0385 | 3486 | 2.8689 | 47 |
| 14 | 2723 | 3.6722 | 2912 | 3.4346 | 3102 | 3.2238 | 3294 | 3.0356 | 3489 | 2.8662 | 46 |
| 15 | 2726 | 3.6680 | 2915 | 3.4308 | 3105 | 3.2205 | 3298 | 3.0326 | 3492 | 2.8636 | 45 |
| 16 | 2729 | 3.6638 | 2918 | 3.4271 | 3108 | 3.2172 | 3301 | 3.0296 | 3495 | 2.8609 | 44 |
| 17 | 2733 | 3.6596 | 2921 | 3.4234 | 3111 | 3.2139 | 3304 | 3.0267 | 3499 | 2.8582 | 43 |
| 18 | 2736 | 3.6554 | 2924 | 3.4197 | 3115 | 3.2106 | 3307 | 3.0237 | 3502 | 2.8556 | 42 |
| 19 | 2739 | 3.6512 | 2927 | 3.4160 | 3118 | 3.2073 | 3310 | 3.0208 | 3505 | 2.8529 | 41 |
| 20 | 2742 | 3.6470 | 2931 | 3.4124 | 3121 | 3.2041 | 3314 | 3.0178 | 3508 | 2.8502 | 40 |
| 21 | 2745 | 3.6429 | 2934 | 3.4087 | 3124 | 3.2008 | 3317 | 3.0149 | 3512 | 2.8476 | 39 |
| 22 | 2748 | 3.6387 | 2937 | 3.4050 | 3127 | 3.1975 | 3320 | 3.0120 | 3515 | 2.8449 | 38 |
| 23 | 2751 | 3.6346 | 2940 | 3.4014 | 3131 | 3.1943 | 3323 | 3.0090 | 3518 | 2.8423 | 37 |
| 24 | 2754 | 3.6305 | 2943 | 3.3977 | 3134 | 3.1910 | 3327 | 3.0061 | 3522 | 2.8397 | 36 |
| 25 | 2758 | 3.6264 | 2946 | 3.3941 | 3137 | 3.1878 | 3330 | 3.0032 | 3525 | 2.8370 | 35 |
| 26 | 2761 | 3.6222 | 2949 | 3.3904 | 3140 | 3.1845 | 3333 | 3.0003 | 3528 | 2.8344 | 34 |
| 27 | 2764 | 3.6181 | 2953 | 3.3868 | 3143 | 3.1813 | 3336 | 2.9974 | 3531 | 2.8318 | 33 |
| 28 | 2767 | 3.6140 | 2956 | 3.3832 | 3147 | 3.1780 | 3339 | 2.9945 | 3535 | 2.8291 | 32 |
| 29 | 2770 | 3.6100 | 2959 | 3.3796 | 3150 | 3.1748 | 3343 | 2.9916 | 3538 | 2.8265 | 31 |
| 30 | 2773 | 3.6059 | 2962 | 3.3759 | 3153 | 3.1716 | 3346 | 2.9887 | 3541 | 2.8239 | 30 |
| 31 | 2776 | 3.6018 | 2965 | 3.3723 | 3156 | 3.1684 | 3349 | 2.9858 | 3544 | 2.8213 | 29 |
| 32 | 2780 | 3.5978 | 2968 | 3.3687 | 3159 | 3.1652 | 3352 | 2.9829 | 3548 | 2.8187 | 28 |
| 33 | 2783 | 3.5937 | 2972 | 3.3652 | 3163 | 3.1620 | 3356 | 2.9800 | 3551 | 2.8161 | 27 |
| 34 | 2786 | 3.5897 | 2975 | 3.3616 | 3166 | 3.1588 | 3359 | 2.9772 | 3554 | 2.8135 | 26 |
| 35 | 2789 | 3.5856 | 2978 | 3.3580 | 3169 | 3.1556 | 3362 | 2.9743 | 3558 | 2.8109 | 25 |
| 36 | 2792 | 3.5816 | 2981 | 3.3544 | 3172 | 3.1524 | 3365 | 2.9714 | 3561 | 2.8083 | 24 |
| 37 | 2795 | 3.5776 | 2984 | 3.3509 | 3175 | 3.1492 | 3369 | 2.9686 | 3564 | 2.8057 | 23 |
| 38 | 2798 | 3.5736 | 2987 | 3.3473 | 3179 | 3.1460 | 3372 | 2.9657 | 3567 | 2.8032 | 22 |
| 39 | 2801 | 3.5696 | 2991 | 3.3438 | 3182 | 3.1429 | 3375 | 2.9629 | 3571 | 2.8006 | 21 |
| 40 | 2805 | 3.5656 | 2994 | 3.3402 | 3185 | 3.1397 | 3378 | 2.9600 | 3574 | 2.7980 | 20 |
| 41 | 2808 | 3.5616 | 2997 | 3.3367 | 3188 | 3.1366 | 3382 | 2.9572 | 3577 | 2.7955 | 19 |
| 42 | 2811 | 3.5576 | 3000 | 3.3332 | 3191 | 3.1334 | 3385 | 2.9544 | 3581 | 2.7929 | 18 |
| 43 | 2814 | 3.5536 | 3003 | 3.3297 | 3195 | 3.1303 | 3388 | 2.9515 | 3584 | 2.7903 | 17 |
| 44 | 2817 | 3.5497 | 3006 | 3.3261 | 3198 | 3.1271 | 3391 | 2.9487 | 3587 | 2.7878 | 16 |
| 45 | 2820 | 3.5457 | 3010 | 3.3226 | 3201 | 3.1240 | 3395 | 2.9459 | 3590 | 2.7852 | 15 |
| 46 | 2823 | 3.5418 | 3013 | 3.3191 | 3204 | 3.1209 | 3398 | 2.9431 | 3594 | 2.7827 | 14 |
| 47 | 2827 | 3.5379 | 3016 | 3.3156 | 3207 | 3.1178 | 3401 | 2.9403 | 3597 | 2.7801 | 13 |
| 48 | 2830 | 3.5339 | 3019 | 3.3122 | 3211 | 3.1146 | 3404 | 2.9375 | 3600 | 2.7776 | 12 |
| 49 | 2833 | 3.5300 | 3022 | 3.3087 | 3214 | 3.1115 | 3408 | 2.9347 | 3604 | 2.7751 | 11 |
| 50 | 2836 | 3.5261 | 3026 | 3.3052 | 3217 | 3.1084 | 3411 | 2.9319 | 3607 | 2.7725 | 10 |
| 51 | 2839 | 3.5222 | 3029 | 3.3017 | 3220 | 3.1053 | 3414 | 2.9291 | 3610 | 2.7700 | 9 |
| 52 | 2842 | 3.5183 | 3032 | 3.2983 | 3223 | 3.1022 | 3417 | 2.9263 | 3613 | 2.7675 | 8 |
| 53 | 2845 | 3.5144 | 3035 | 3.2948 | 3227 | 3.0991 | 3421 | 2.9235 | 3617 | 2.7650 | 7 |
| 54 | 2849 | 3.5105 | 3038 | 3.2914 | 3230 | 3.0961 | 3424 | 2.9208 | 3620 | 2.7625 | 6 |
| 55 | 2852 | 3.5067 | 3041 | 3.2880 | 3233 | 3.0930 | 3427 | 2.9180 | 3623 | 2.7600 | 5 |
| 56 | 2855 | 3.5028 | 3045 | 3.2845 | 3236 | 3.0899 | 3430 | 2.9152 | 3627 | 2.7575 | 4 |
| 57 | 2858 | 3.4989 | 3048 | 3.2811 | 3240 | 3.0868 | 3434 | 2.9125 | 3630 | 2.7550 | 3 |
| 58 | 2861 | 3.4951 | 3051 | 3.2777 | 3243 | 3.0838 | 3437 | 2.9097 | 3633 | 2.7525 | 2 |
| 59 | 2864 | 3.4912 | 3054 | 3.2743 | 3246 | 3.0807 | 3440 | 2.9070 | 3636 | 2.7500 | 1 |
| 60 | 2867 | 3.4874 | 3057 | 3.2709 | 3249 | 3.0777 | 3443 | 2.9042 | 3640 | 2.7475 | 0 |
| | cot | tan | cot | tan | cot | tan | cot | tan | cot | tan | |
| ' | 74° | | 73° | | 72° | | 71° | | 70° | | ' |

| ′ | 20° | | 21° | | 22° | | 23° | | 24° | | ′ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | tan | cot | tan | cot | tan | cot | tan | cot | tan | cot | |
| **0** | 3640 | 2.7475 | 3839 | 2.6051 | 4040 | 2.4751 | 4245 | 2.3559 | 4452 | 2.2460 | **60** |
| 1 | 3643 | 2.7450 | 3842 | 2.6028 | 4044 | 2.4730 | 4248 | 2.3539 | 4456 | 2.2443 | 59 |
| 2 | 3646 | 2.7425 | 3845 | 2.6006 | 4047 | 2.4709 | 4252 | 2.3520 | 4459 | 2.2425 | 58 |
| 3 | 3650 | 2.7400 | 3849 | 2.5983 | 4050 | 2.4689 | 4255 | 2.3501 | 4463 | 2.2408 | 57 |
| 4 | 3653 | 2.7376 | 3852 | 2.5961 | 4054 | 2.4668 | 4258 | 2.3483 | 4466 | 2.2390 | 56 |
| **5** | 3656 | 2.7351 | 3855 | 2.5938 | 4057 | 2.4648 | 4262 | 2.3464 | 4470 | 2.2373 | **55** |
| 6 | 3659 | 2.7326 | 3859 | 2.5916 | 4061 | 2.4627 | 4265 | 2.3445 | 4473 | 2.2355 | 54 |
| 7 | 3663 | 2.7302 | 3862 | 2.5893 | 4064 | 2.4606 | 4269 | 2.3426 | 4477 | 2.2338 | 53 |
| 8 | 3666 | 2.7277 | 3865 | 2.5871 | 4067 | 2.4586 | 4272 | 2.3407 | 4480 | 2.2320 | 52 |
| 9 | 3669 | 2.7253 | 3869 | 2.5848 | 4071 | 2.4566 | 4276 | 2.3388 | 4484 | 2.2303 | 51 |
| **10** | 3673 | 2.7228 | 3872 | 2.5826 | 4074 | 2.4545 | 4279 | 2.3369 | 4487 | 2.2286 | **50** |
| 11 | 3676 | 2.7204 | 3875 | 2.5804 | 4078 | 2.4525 | 4283 | 2.3351 | 4491 | 2.2268 | 49 |
| 12 | 3679 | 2.7179 | 3879 | 2.5782 | 4081 | 2.4504 | 4286 | 2.3332 | 4494 | 2.2251 | 48 |
| 13 | 3683 | 2.7155 | 3882 | 2.5759 | 4084 | 2.4484 | 4289 | 2.3313 | 4498 | 2.2234 | 47 |
| 14 | 3686 | 2.7130 | 3885 | 2.5737 | 4088 | 2.4464 | 4293 | 2.3294 | 4501 | 2.2216 | 46 |
| **15** | 3689 | 2.7106 | 3889 | 2.5715 | 4091 | 2.4443 | 4296 | 2.3276 | 4505 | 2.2199 | **45** |
| 16 | 3693 | 2.7082 | 3892 | 2.5693 | 4095 | 2.4423 | 4300 | 2.3257 | 4508 | 2.2182 | 44 |
| 17 | 3696 | 2.7058 | 3895 | 2.5671 | 4098 | 2.4403 | 4303 | 2.3238 | 4512 | 2.2165 | 43 |
| 18 | 3699 | 2.7034 | 3899 | 2.5649 | 4101 | 2.4383 | 4307 | 2.3220 | 4515 | 2.2148 | 42 |
| 19 | 3702 | 2.7009 | 3902 | 2.5627 | 4105 | 2.4362 | 4310 | 2.3201 | 4519 | 2.2130 | 41 |
| **20** | 3706 | 2.6985 | 3906 | 2.5605 | 4108 | 2.4342 | 4314 | 2.3183 | 4522 | 2.2113 | **40** |
| 21 | 3709 | 2.6961 | 3909 | 2.5533 | 4111 | 2.4322 | 4317 | 2.3164 | 4526 | 2.2096 | 39 |
| 22 | 3712 | 2.6937 | 3912 | 2.5561 | 4115 | 2.4302 | 4320 | 2.3146 | 4529 | 2.2079 | 38 |
| 23 | 3716 | 2.6913 | 3916 | 2.5539 | 4118 | 2.4282 | 4324 | 2.3127 | 4533 | 2.2062 | 37 |
| 24 | 3719 | 2.6889 | 3919 | 2.5517 | 4122 | 2.4262 | 4327 | 2.3109 | 4536 | 2.2045 | 36 |
| **25** | 3722 | 2.6865 | 3922 | 2.5495 | 4125 | 2.4242 | 4331 | 2.3090 | 4540 | 2.2028 | **35** |
| 26 | 3726 | 2.6841 | 3926 | 2.5473 | 4129 | 2.4222 | 4334 | 2.3072 | 4543 | 2.2011 | 34 |
| 27 | 3729 | 2.6818 | 3929 | 2.5452 | 4132 | 2.4202 | 4338 | 2.3053 | 4547 | 2.1994 | 33 |
| 28 | 3732 | 2.6794 | 3932 | 2.5430 | 4135 | 2.4182 | 4341 | 2.3035 | 4550 | 2.1977 | 32 |
| 29 | 3736 | 2.6770 | 3936 | 2.5408 | 4139 | 2.4162 | 4345 | 2.3017 | 4554 | 2.1960 | 31 |
| **30** | 3739 | 2.6746 | 3939 | 2.5386 | 4142 | 2.4142 | 4348 | 2.2998 | 4557 | 2.1943 | **30** |
| 31 | 3742 | 2.6723 | 3942 | 2.5365 | 4146 | 2.4122 | 4352 | 2.2980 | 4561 | 2.1926 | 29 |
| 32 | 3745 | 2.6699 | 3946 | 2.5343 | 4149 | 2.4102 | 4355 | 2.2962 | 4564 | 2.1909 | 28 |
| 33 | 3749 | 2.6675 | 3949 | 2.5322 | 4152 | 2.4083 | 4359 | 2.2944 | 4568 | 2.1892 | 27 |
| 34 | 3752 | 2.6652 | 3953 | 2.5300 | 4156 | 2.4063 | 4362 | 2.2925 | 4571 | 2.1876 | 26 |
| **35** | 3755 | 2.6628 | 3956 | 2.5279 | 4159 | 2.4043 | 4365 | 2.2907 | 4575 | 2.1859 | **25** |
| 36 | 3759 | 2.6605 | 3959 | 2.5257 | 4163 | 2.4023 | 4369 | 2.2889 | 4578 | 2.1842 | 24 |
| 37 | 3762 | 2.6581 | 3963 | 2.5236 | 4166 | 2.4004 | 4372 | 2.2871 | 4582 | 2.1825 | 23 |
| 38 | 3765 | 2.6558 | 3966 | 2.5214 | 4169 | 2.3984 | 4376 | 2.2853 | 4585 | 2.1808 | 22 |
| 39 | 3769 | 2.6534 | 3969 | 2.5193 | 4173 | 2.3964 | 4379 | 2.2835 | 4589 | 2.1792 | 21 |
| **40** | 3772 | 2.6511 | 3973 | 2.5172 | 4176 | 2.3945 | 4383 | 2.2817 | 4592 | 2.1775 | **20** |
| 41 | 3775 | 2.6488 | 3976 | 2.5150 | 4180 | 2.3925 | 4386 | 2.2799 | 4596 | 2.1758 | 19 |
| 42 | 3779 | 2.6464 | 3979 | 2.5129 | 4183 | 2.3906 | 4390 | 2.2781 | 4599 | 2.1742 | 18 |
| 43 | 3782 | 2.6441 | 3983 | 2.5108 | 4187 | 2.3886 | 4393 | 2.2763 | 4603 | 2.1725 | 17 |
| 44 | 3785 | 2.6418 | 3986 | 2.5086 | 4190 | 2.3867 | 4397 | 2.2745 | 4607 | 2.1708 | 16 |
| **45** | 3789 | 2.6395 | 3990 | 2.5065 | 4193 | 2.3847 | 4400 | 2.2727 | 4610 | 2.1692 | **15** |
| 46 | 3792 | 2.6371 | 3993 | 2.5044 | 4197 | 2.3828 | 4404 | 2.2709 | 4614 | 2.1675 | 14 |
| 47 | 3795 | 2.6348 | 3996 | 2.5023 | 4200 | 2.3808 | 4407 | 2.2691 | 4617 | 2.1659 | 13 |
| 48 | 3799 | 2.6325 | 4000 | 2.5002 | 4204 | 2.3789 | 4411 | 2.2673 | 4621 | 2.1642 | 12 |
| 49 | 3802 | 2.6302 | 4003 | 2.4981 | 4207 | 2.3770 | 4414 | 2.2655 | 4624 | 2.1625 | 11 |
| **50** | 3805 | 2.6279 | 4006 | 2.4960 | 4210 | 2.3750 | 4417 | 2.2637 | 4628 | 2.1609 | **10** |
| 51 | 3809 | 2.6256 | 4010 | 2.4939 | 4214 | 2.3731 | 4421 | 2.2620 | 4631 | 2.1592 | 9 |
| 52 | 3812 | 2.6233 | 4013 | 2.4918 | 4217 | 2.3712 | 4424 | 2.2602 | 4635 | 2.1576 | 8 |
| 53 | 3815 | 2.6210 | 4017 | 2.4897 | 4221 | 2.3693 | 4428 | 2.2584 | 4638 | 2.1560 | 7 |
| 54 | 3819 | 2.6187 | 4020 | 2.4876 | 4224 | 2.3673 | 4431 | 2.2566 | 4642 | 2.1543 | 6 |
| **55** | 3822 | 2.6165 | 4023 | 2.4855 | 4228 | 2.3654 | 4435 | 2.2549 | 4645 | 2.1527 | **5** |
| 56 | 3825 | 2.6142 | 4027 | 2.4834 | 4231 | 2.3635 | 4438 | 2.2531 | 4649 | 2.1510 | 4 |
| 57 | 3829 | 2.6119 | 4030 | 2.4813 | 4234 | 2.3616 | 4442 | 2.2513 | 4652 | 2.1494 | 3 |
| 58 | 3832 | 2.6096 | 4033 | 2.4792 | 4238 | 2.3597 | 4445 | 2.2496 | 4656 | 2.1478 | 2 |
| 59 | 3835 | 2.6074 | 4037 | 2.4772 | 4241 | 2.3578 | 4449 | 2.2478 | 4660 | 2.1461 | 1 |
| **60** | 3839 | 2.6051 | 4040 | 2.4751 | 4245 | 2.3559 | 4452 | 2.2460 | 4663 | 2.1445 | **0** |
| | cot | tan | cot | tan | cot | tan | cot | tan | cot | tan | |
| ′ | 69° | | 68° | | 67° | | 66° | | 65° | | ′ |

| ′ | 25° | | 26° | | 27° | | 28° | | 29° | | ′ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | tan | cot | tan | cot | tan | cot | tan | cot | tan | cot | |
| 0 | 4663 | 2.1445 | 4877 | 2.0503 | 5095 | 1.9626 | 5317 | 1.8807 | 5543 | 1.8040 | 60 |
| 1 | 4667 | 2.1429 | 4881 | 2.0488 | 5099 | 1.9612 | 5321 | 1.8794 | 5547 | 1.8028 | 59 |
| 2 | 4670 | 2.1413 | 4885 | 2.0473 | 5103 | 1.9598 | 5325 | 1.8781 | 5551 | 1.8016 | 58 |
| 3 | 4674 | 2.1396 | 4888 | 2.0458 | 5106 | 1.9584 | 5328 | 1.8768 | 5555 | 1.8003 | 57 |
| 4 | 4677 | 2.1380 | 4892 | 2.0443 | 5110 | 1.9570 | 5332 | 1.8755 | 5558 | 1.7991 | 56 |
| 5 | 4681 | 2.1364 | 4895 | 2.0428 | 5114 | 1.9556 | 5336 | 1.8741 | 5562 | 1.7979 | 55 |
| 6 | 4684 | 2.1348 | 4899 | 2.0413 | 5117 | 1.9542 | 5340 | 1.8728 | 5566 | 1.7966 | 54 |
| 7 | 4688 | 2.1332 | 4903 | 2.0398 | 5121 | 1.9528 | 5343 | 1.8715 | 5570 | 1.7954 | 53 |
| 8 | 4691 | 2.1315 | 4906 | 2.0383 | 5125 | 1.9514 | 5347 | 1.8702 | 5574 | 1.7942 | 52 |
| 9 | 4695 | 2.1299 | 4910 | 2.0368 | 5128 | 1.9500 | 5351 | 1.8689 | 5577 | 1.7930 | 51 |
| 10 | 4699 | 2.1283 | 4913 | 2.0353 | 5132 | 1.9486 | 5354 | 1.8676 | 5581 | 1.7917 | 50 |
| 11 | 4702 | 2.1267 | 4917 | 2.0338 | 5136 | 1.9472 | 5358 | 1.8663 | 5585 | 1.7905 | 49 |
| 12 | 4706 | 2.1251 | 4921 | 2.0323 | 5139 | 1.9458 | 5362 | 1.8650 | 5589 | 1.7893 | 48 |
| 13 | 4709 | 2.1235 | 4924 | 2.0308 | 5143 | 1.9444 | 5366 | 1.8637 | 5593 | 1.7881 | 47 |
| 14 | 4713 | 2.1219 | 4928 | 2.0293 | 5147 | 1.9430 | 5369 | 1.8624 | 5596 | 1.7868 | 46 |
| 15 | 4716 | 2.1203 | 4931 | 2.0278 | 5150 | 1.9416 | 5373 | 1.8611 | 5600 | 1.7856 | 45 |
| 16 | 4720 | 2.1187 | 4935 | 2.0263 | 5154 | 1.9402 | 5377 | 1.8598 | 5604 | 1.7844 | 44 |
| 17 | 4723 | 2.1171 | 4939 | 2.0248 | 5158 | 1.9388 | 5381 | 1.8585 | 5608 | 1.7832 | 43 |
| 18 | 4727 | 2.1155 | 4942 | 2.0233 | 5161 | 1.9375 | 5384 | 1.8572 | 5612 | 1.7820 | 42 |
| 19 | 4731 | 2.1139 | 4946 | 2.0219 | 5165 | 1.9361 | 5388 | 1.8559 | 5616 | 1.7808 | 41 |
| 20 | 4734 | 2.1123 | 4950 | 2.0204 | 5169 | 1.9347 | 5392 | 1.8546 | 5619 | 1.7796 | 40 |
| 21 | 4738 | 2.1107 | 4953 | 2.0189 | 5172 | 1.9333 | 5396 | 1.8533 | 5623 | 1.7783 | 39 |
| 22 | 4741 | 2.1092 | 4957 | 2.0174 | 5176 | 1.9319 | 5399 | 1.8520 | 5627 | 1.7771 | 38 |
| 23 | 4745 | 2.1076 | 4960 | 2.0160 | 5180 | 1.9306 | 5403 | 1.8507 | 5631 | 1.7759 | 37 |
| 24 | 4748 | 2.1060 | 4964 | 2.0145 | 5184 | 1.9292 | 5407 | 1.8495 | 5635 | 1.7747 | 36 |
| 25 | 4752 | 2.1044 | 4968 | 2.0130 | 5187 | 1.9278 | 5411 | 1.8482 | 5639 | 1.7735 | 35 |
| 26 | 4755 | 2.1028 | 4971 | 2.0115 | 5191 | 1.9265 | 5415 | 1.8469 | 5642 | 1.7723 | 34 |
| 27 | 4759 | 2.1013 | 4975 | 2.0101 | 5195 | 1.9251 | 5418 | 1.8456 | 5646 | 1.7711 | 33 |
| 28 | 4763 | 2.0997 | 4979 | 2.0086 | 5198 | 1.9237 | 5422 | 1.8443 | 5650 | 1.7699 | 32 |
| 29 | 4766 | 2.0981 | 4982 | 2.0072 | 5202 | 1.9223 | 5426 | 1.8430 | 5654 | 1.7687 | 31 |
| 30 | 4770 | 2.0965 | 4986 | 2.0057 | 5206 | 1.9210 | 5430 | 1.8418 | 5658 | 1.7675 | 30 |
| 31 | 4773 | 2.0950 | 4989 | 2.0042 | 5209 | 1.9196 | 5433 | 1.8405 | 5662 | 1.7663 | 29 |
| 32 | 4777 | 2.0934 | 4993 | 2.0028 | 5213 | 1.9183 | 5437 | 1.8392 | 5665 | 1.7651 | 28 |
| 33 | 4780 | 2.0918 | 4997 | 2.0013 | 5217 | 1.9169 | 5441 | 1.8379 | 5669 | 1.7639 | 27 |
| 34 | 4784 | 2.0903 | 5000 | 1.9999 | 5220 | 1.9155 | 5445 | 1.8367 | 5673 | 1.7627 | 26 |
| 35 | 4788 | 2.0887 | 5004 | 1.9984 | 5224 | 1.9142 | 5448 | 1.8354 | 5677 | 1.7615 | 25 |
| 36 | 4791 | 2.0872 | 5008 | 1.9970 | 5228 | 1.9128 | 5452 | 1.8341 | 5681 | 1.7603 | 24 |
| 37 | 4795 | 2.0856 | 5011 | 1.9955 | 5232 | 1.9115 | 5456 | 1.8329 | 5685 | 1.7591 | 23 |
| 38 | 4798 | 2.0840 | 5015 | 1.9941 | 5235 | 1.9101 | 5460 | 1.8316 | 5688 | 1.7579 | 22 |
| 39 | 4802 | 2.0825 | 5019 | 1.9926 | 5239 | 1.9088 | 5464 | 1.8303 | 5692 | 1.7567 | 21 |
| 40 | 4806 | 2.0809 | 5022 | 1.9912 | 5243 | 1.9074 | 5467 | 1.8291 | 5696 | 1.7556 | 20 |
| 41 | 4809 | 2.0794 | 5026 | 1.9897 | 5246 | 1.9061 | 5471 | 1.8278 | 5700 | 1.7544 | 19 |
| 42 | 4813 | 2.0778 | 5029 | 1.9883 | 5250 | 1.9047 | 5475 | 1.8265 | 5704 | 1.7532 | 18 |
| 43 | 4816 | 2.0763 | 5033 | 1.9868 | 5254 | 1.9034 | 5479 | 1.8253 | 5708 | 1.7520 | 17 |
| 44 | 4820 | 2.0748 | 5037 | 1.9854 | 5258 | 1.9020 | 5482 | 1.8240 | 5712 | 1.7508 | 16 |
| 45 | 4823 | 2.0732 | 5040 | 1.9840 | 5261 | 1.9007 | 5486 | 1.8228 | 5715 | 1.7496 | 15 |
| 46 | 4827 | 2.0717 | 5044 | 1.9825 | 5265 | 1.8993 | 5490 | 1.8215 | 5719 | 1.7485 | 14 |
| 47 | 4831 | 2.0701 | 5048 | 1.9811 | 5269 | 1.8980 | 5494 | 1.8202 | 5723 | 1.7473 | 13 |
| 48 | 4834 | 2.0686 | 5051 | 1.9797 | 5272 | 1.8967 | 5498 | 1.8190 | 5727 | 1.7461 | 12 |
| 49 | 4838 | 2.0671 | 5055 | 1.9782 | 5276 | 1.8953 | 5501 | 1.8177 | 5731 | 1.7449 | 11 |
| 50 | 4841 | 2.0655 | 5059 | 1.9768 | 5280 | 1.8940 | 5505 | 1.8165 | 5735 | 1.7437 | 10 |
| 51 | 4845 | 2.0640 | 5062 | 1.9754 | 5284 | 1.8927 | 5509 | 1.8152 | 5739 | 1.7426 | 9 |
| 52 | 4849 | 2.0625 | 5066 | 1.9740 | 5287 | 1.8913 | 5513 | 1.8140 | 5743 | 1.7414 | 8 |
| 53 | 4852 | 2.0609 | 5070 | 1.9725 | 5291 | 1.8900 | 5517 | 1.8127 | 5746 | 1.7402 | 7 |
| 54 | 4856 | 2.0594 | 5073 | 1.9711 | 5295 | 1.8887 | 5520 | 1.8115 | 5750 | 1.7391 | 6 |
| 55 | 4859 | 2.0579 | 5077 | 1.9697 | 5298 | 1.8873 | 5524 | 1.8103 | 5754 | 1.7379 | 5 |
| 56 | 4863 | 2.0564 | 5081 | 1.9683 | 5302 | 1.8860 | 5528 | 1.8090 | 5758 | 1.7367 | 4 |
| 57 | 4867 | 2.0549 | 5084 | 1.9669 | 5306 | 1.8847 | 5532 | 1.8078 | 5762 | 1.7355 | 3 |
| 58 | 4870 | 2.0533 | 5088 | 1.9654 | 5310 | 1.8834 | 5535 | 1.8065 | 5766 | 1.7344 | 2 |
| 59 | 4874 | 2.0518 | 5092 | 1.9640 | 5313 | 1.8820 | 5539 | 1.8053 | 5770 | 1.7332 | 1 |
| 60 | 4877 | 2.0503 | 5095 | 1.9626 | 5317 | 1.8807 | 5543 | 1.8040 | 5774 | 1.7321 | 0 |
| | cot | tan | cot | tan | cot | tan | cot | tan | cot | tan | |
| ′ | 64° | | 63° | | 62° | | 61° | | 60° | | ′ |

# NATURAL TANGENTS AND COTANGENTS

| ′ | 30° | | 31° | | 32° | | 33° | | 34° | | ′ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | tan | cot | tan | cot | tan | cot | tan | cot | tan | cot | |
| **0** | 5774 | 1.7321 | 6009 | 1.6643 | 6249 | 1.6003 | 6494 | 1.5399 | 6745 | 1.4826 | **60** |
| 1 | 5777 | 1.7309 | 6013 | 1.6632 | 6253 | 1.5993 | 6498 | 1.5389 | 6749 | 1.4816 | 59 |
| 2 | 5781 | 1.7297 | 6017 | 1.6621 | 6257 | 1.5983 | 6502 | 1.5379 | 6754 | 1.4807 | 58 |
| 3 | 5785 | 1.7286 | 6020 | 1.6610 | 6261 | 1.5972 | 6506 | 1.5369 | 6758 | 1.4798 | 57 |
| 4 | 5789 | 1.7274 | 6024 | 1.6599 | 6265 | 1.5962 | 6511 | 1.5359 | 6762 | 1.4788 | 56 |
| **5** | 5793 | 1.7262 | 6028 | 1.6588 | 6269 | 1.5952 | 6515 | 1.5350 | 6766 | 1.4779 | **55** |
| 6 | 5797 | 1.7251 | 6032 | 1.6577 | 6273 | 1.5941 | 6519 | 1.5340 | 6771 | 1.4770 | 54 |
| 7 | 5801 | 1.7239 | 6036 | 1.6566 | 6277 | 1.5931 | 6523 | 1.5330 | 6775 | 1.4761 | 53 |
| 8 | 5805 | 1.7228 | 6040 | 1.6555 | 6281 | 1.5921 | 6527 | 1.5320 | 6779 | 1.4751 | 52 |
| 9 | 5808 | 1.7216 | 6044 | 1.6545 | 6285 | 1.5911 | 6531 | 1.5311 | 6783 | 1.4742 | 51 |
| **10** | 5812 | 1.7205 | 6048 | 1.6534 | 6289 | 1.5900 | 6536 | 1.5301 | 6787 | 1.4733 | **50** |
| 11 | 5816 | 1.7193 | 6052 | 1.6523 | 6293 | 1.5890 | 6540 | 1.5291 | 6792 | 1.4724 | 49 |
| 12 | 5820 | 1.7182 | 6056 | 1.6512 | 6297 | 1.5880 | 6544 | 1.5282 | 6796 | 1.4715 | 48 |
| 13 | 5824 | 1.7170 | 6060 | 1.6501 | 6301 | 1.5869 | 6548 | 1.5272 | 6800 | 1.4705 | 47 |
| 14 | 5828 | 1.7159 | 6064 | 1.6490 | 6305 | 1.5859 | 6552 | 1.5262 | 6805 | 1.4696 | 46 |
| **15** | 5832 | 1.7147 | 6068 | 1.6479 | 6310 | 1.5849 | 6556 | 1.5253 | 6809 | 1.4687 | **45** |
| 16 | 5836 | 1.7136 | 6072 | 1.6469 | 6314 | 1.5839 | 6560 | 1.5243 | 6813 | 1.4678 | 44 |
| 17 | 5840 | 1.7124 | 6076 | 1.6458 | 6318 | 1.5829 | 6565 | 1.5233 | 6817 | 1.4669 | 43 |
| 18 | 5844 | 1.7113 | 6080 | 1.6447 | 6322 | 1.5818 | 6569 | 1.5224 | 6822 | 1.4659 | 42 |
| 19 | 5847 | 1.7102 | 6084 | 1.6436 | 6326 | 1.5808 | 6573 | 1.5214 | 6826 | 1.4650 | 41 |
| **20** | 5851 | 1.7090 | 6088 | 1.6426 | 6330 | 1.5798 | 6577 | 1.5204 | 6830 | 1.4641 | **40** |
| 21 | 5855 | 1.7079 | 6092 | 1.6415 | 6334 | 1.5788 | 6581 | 1.5195 | 6834 | 1.4632 | 39 |
| 22 | 5859 | 1.7067 | 6096 | 1.6404 | 6338 | 1.5778 | 6585 | 1.5185 | 6839 | 1.4623 | 38 |
| 23 | 5863 | 1.7056 | 6100 | 1.6393 | 6342 | 1.5768 | 6590 | 1.5175 | 6843 | 1.4614 | 37 |
| 24 | 5867 | 1.7045 | 6104 | 1.6383 | 6346 | 1.5757 | 6594 | 1.5166 | 6847 | 1.4605 | 36 |
| **25** | 5871 | 1.7033 | 6108 | 1.6372 | 6350 | 1.5747 | 6598 | 1.5156 | 6851 | 1.4596 | **35** |
| 26 | 5875 | 1.7022 | 6112 | 1.6361 | 6354 | 1.5737 | 6602 | 1.5147 | 6856 | 1.4586 | 34 |
| 27 | 5879 | 1.7011 | 6116 | 1.6351 | 6358 | 1.5727 | 6606 | 1.5137 | 6860 | 1.4577 | 33 |
| 28 | 5883 | 1.6999 | 6120 | 1.6340 | 6363 | 1.5717 | 6610 | 1.5127 | 6864 | 1.4568 | 32 |
| 29 | 5887 | 1.6988 | 6124 | 1.6329 | 6367 | 1.5707 | 6615 | 1.5118 | 6869 | 1.4559 | 31 |
| **30** | 5890 | 1.6977 | 6128 | 1.6319 | 6371 | 1.5697 | 6619 | 1.5108 | 6873 | 1.4550 | **30** |
| 31 | 5894 | 1.6965 | 6132 | 1.6308 | 6375 | 1.5687 | 6623 | 1.5099 | 6877 | 1.4541 | 29 |
| 32 | 5898 | 1.6954 | 6136 | 1.6297 | 6379 | 1.5677 | 6627 | 1.5089 | 6881 | 1.4532 | 28 |
| 33 | 5902 | 1.6943 | 6140 | 1.6287 | 6383 | 1.5667 | 6631 | 1.5080 | 6886 | 1.4523 | 27 |
| 34 | 5906 | 1.6932 | 6144 | 1.6276 | 6387 | 1.5657 | 6636 | 1.5070 | 6890 | 1.4514 | 26 |
| **35** | 5910 | 1.6920 | 6148 | 1.6265 | 6391 | 1.5647 | 6640 | 1.5061 | 6894 | 1.4505 | **25** |
| 36 | 5914 | 1.6909 | 6152 | 1.6255 | 6395 | 1.5637 | 6644 | 1.5051 | 6899 | 1.4496 | 24 |
| 37 | 5918 | 1.6898 | 6156 | 1.6244 | 6399 | 1.5627 | 6648 | 1.5042 | 6903 | 1.4487 | 23 |
| 38 | 5922 | 1.6887 | 6160 | 1.6234 | 6403 | 1.5617 | 6652 | 1.5032 | 6907 | 1.4478 | 22 |
| 39 | 5926 | 1.6875 | 6164 | 1.6223 | 6408 | 1.5607 | 6657 | 1.5023 | 6911 | 1.4469 | 21 |
| **40** | 5930 | 1.6864 | 6168 | 1.6212 | 6412 | 1.5597 | 6661 | 1.5013 | 6916 | 1.4460 | **20** |
| 41 | 5934 | 1.6853 | 6172 | 1.6202 | 6416 | 1.5587 | 6665 | 1.5004 | 6920 | 1.4451 | 19 |
| 42 | 5938 | 1.6842 | 6176 | 1.6191 | 6420 | 1.5577 | 6669 | 1.4994 | 6924 | 1.4442 | 18 |
| 43 | 5942 | 1.6831 | 6180 | 1.6181 | 6424 | 1.5567 | 6673 | 1.4985 | 6929 | 1.4433 | 17 |
| 44 | 5945 | 1.6820 | 6184 | 1.6170 | 6428 | 1.5557 | 6678 | 1.4975 | 6933 | 1.4424 | 16 |
| **45** | 5949 | 1.6808 | 6188 | 1.6160 | 6432 | 1.5547 | 6682 | 1.4966 | 6937 | 1.4415 | **15** |
| 46 | 5953 | 1.6797 | 6192 | 1.6149 | 6436 | 1.5537 | 6686 | 1.4957 | 6942 | 1.4406 | 14 |
| 47 | 5957 | 1.6786 | 6196 | 1.6139 | 6440 | 1.5527 | 6690 | 1.4947 | 6946 | 1.4397 | 13 |
| 48 | 5961 | 1.6775 | 6200 | 1.6128 | 6445 | 1.5517 | 6694 | 1.4938 | 6950 | 1.4388 | 12 |
| 49 | 5965 | 1.6764 | 6204 | 1.6118 | 6449 | 1.5507 | 6699 | 1.4928 | 6954 | 1.4379 | 11 |
| **50** | 5969 | 1.6753 | 6208 | 1.6107 | 6453 | 1.5497 | 6703 | 1.4919 | 6959 | 1.4370 | **10** |
| 51 | 5973 | 1.6742 | 6212 | 1.6097 | 6457 | 1.5487 | 6707 | 1.4910 | 6963 | 1.4361 | 9 |
| 52 | 5977 | 1.6731 | 6216 | 1.6087 | 6461 | 1.5477 | 6711 | 1.4900 | 6967 | 1.4352 | 8 |
| 53 | 5981 | 1.6720 | 6220 | 1.6076 | 6465 | 1.5468 | 6716 | 1.4891 | 6972 | 1.4344 | 7 |
| 54 | 5985 | 1.6709 | 6224 | 1.6066 | 6469 | 1.5458 | 6720 | 1.4882 | 6976 | 1.4335 | 6 |
| **55** | 5989 | 1.6698 | 6228 | 1.6055 | 6473 | 1.5448 | 6724 | 1.4872 | 6980 | 1.4326 | **5** |
| 56 | 5993 | 1.6687 | 6233 | 1.6045 | 6478 | 1.5438 | 6728 | 1.4863 | 6985 | 1.4317 | 4 |
| 57 | 5997 | 1.6676 | 6237 | 1.6034 | 6482 | 1.5428 | 6732 | 1.4854 | 6989 | 1.4308 | 3 |
| 58 | 6001 | 1.6665 | 6241 | 1.6024 | 6486 | 1.5418 | 6737 | 1.4844 | 6993 | 1.4299 | 2 |
| 59 | 6005 | 1.6654 | 6245 | 1.6014 | 6490 | 1.5408 | 6741 | 1.4835 | 6998 | 1.4290 | 1 |
| **60** | 6009 | 1.6643 | 6249 | 1.6003 | 6494 | 1.5399 | 6745 | 1.4826 | 7002 | 1.4281 | **0** |
| | cot | tan | cot | tan | cot | tan | cot | tan | cot | tan | |
| ′ | 59° | | 58° | | 57° | | 56° | | 55° | | ′ |

| ′ | 35° | | 36° | | 37° | | 38° | | 39° | | ′ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | tan | cot | tan | cot | tan | cot | tan | cot | tan | cot | |
| 0 | 7002 | 1.4281 | 7265 | 1.3764 | 7536 | 1.3270 | 7813 | 1.2799 | 8098 | 1.2349 | 60 |
| 1 | 7006 | 1.4273 | 7270 | 1.3755 | 7540 | 1.3262 | 7818 | 1.2792 | 8103 | 1.2342 | 59 |
| 2 | 7011 | 1.4264 | 7274 | 1.3747 | 7545 | 1.3254 | 7822 | 1.2784 | 8107 | 1.2334 | 58 |
| 3 | 7015 | 1.4255 | 7279 | 1.3739 | 7549 | 1.3246 | 7827 | 1.2776 | 8112 | 1.2327 | 57 |
| 4 | 7019 | 1.4246 | 7283 | 1.3730 | 7554 | 1.3238 | 7832 | 1.2769 | 8117 | 1.2320 | 56 |
| 5 | 7024 | 1.4237 | 7288 | 1.3722 | 7558 | 1.3230 | 7836 | 1.2761 | 8122 | 1.2312 | 55 |
| 6 | 7028 | 1.4229 | 7292 | 1.3713 | 7563 | 1.3222 | 7841 | 1.2753 | 8127 | 1.2305 | 54 |
| 7 | 7032 | 1.4220 | 7297 | 1.3705 | 7568 | 1.3214 | 7846 | 1.2746 | 8132 | 1.2298 | 53 |
| 8 | 7037 | 1.4211 | 7301 | 1.3697 | 7572 | 1.3206 | 7850 | 1.2738 | 8136 | 1.2290 | 52 |
| 9 | 7041 | 1.4202 | 7306 | 1.3688 | 7577 | 1.3198 | 7855 | 1.2731 | 8141 | 1.2283 | 51 |
| 10 | 7046 | 1.4193 | 7310 | 1.3680 | 7581 | 1.3190 | 7860 | 1.2723 | 8146 | 1.2276 | 50 |
| 11 | 7050 | 1.4185 | 7314 | 1.3672 | 7586 | 1.3182 | 7865 | 1.2715 | 8151 | 1.2268 | 49 |
| 12 | 7054 | 1.4176 | 7319 | 1.3663 | 7590 | 1.3175 | 7869 | 1.2708 | 8156 | 1.2261 | 48 |
| 13 | 7059 | 1.4167 | 7323 | 1.3655 | 7595 | 1.3167 | 7874 | 1.2700 | 8161 | 1.2254 | 47 |
| 14 | 7063 | 1.4158 | 7328 | 1.3647 | 7600 | 1.3159 | 7879 | 1.2693 | 8165 | 1.2247 | 46 |
| 15 | 7067 | 1.4150 | 7332 | 1.3638 | 7604 | 1.3151 | 7883 | 1.2685 | 8170 | 1.2239 | 45 |
| 16 | 7072 | 1.4141 | 7337 | 1.3630 | 7609 | 1.3143 | 7888 | 1.2677 | 8175 | 1.2232 | 44 |
| 17 | 7076 | 1.4132 | 7341 | 1.3622 | 7613 | 1.3135 | 7893 | 1.2670 | 8180 | 1.2225 | 43 |
| 18 | 7080 | 1.4124 | 7346 | 1.3613 | 7618 | 1.3127 | 7898 | 1.2662 | 8185 | 1.2218 | 42 |
| 19 | 7085 | 1.4115 | 7350 | 1.3605 | 7623 | 1.3119 | 7902 | 1.2655 | 8190 | 1.2210 | 41 |
| 20 | 7089 | 1.4106 | 7355 | 1.3597 | 7627 | 1.3111 | 7907 | 1.2647 | 8195 | 1.2203 | 40 |
| 21 | 7094 | 1.4097 | 7359 | 1.3588 | 7632 | 1.3103 | 7912 | 1.2640 | 8199 | 1.2196 | 39 |
| 22 | 7098 | 1.4089 | 7364 | 1.3580 | 7636 | 1.3095 | 7916 | 1.2632 | 8204 | 1.2189 | 38 |
| 23 | 7102 | 1.4080 | 7368 | 1.3572 | 7641 | 1.3087 | 7921 | 1.2624 | 8209 | 1.2181 | 37 |
| 24 | 7107 | 1.4071 | 7373 | 1.3564 | 7646 | 1.3079 | 7926 | 1.2617 | 8214 | 1.2174 | 36 |
| 25 | 7111 | 1.4063 | 7377 | 1.3555 | 7650 | 1.3072 | 7931 | 1.2609 | 8219 | 1.2167 | 35 |
| 26 | 7115 | 1.4054 | 7382 | 1.3547 | 7655 | 1.3064 | 7935 | 1.2602 | 8224 | 1.2160 | 34 |
| 27 | 7120 | 1.4045 | 7386 | 1.3539 | 7659 | 1.3056 | 7940 | 1.2594 | 8229 | 1.2153 | 33 |
| 28 | 7124 | 1.4037 | 7391 | 1.3531 | 7664 | 1.3048 | 7945 | 1.2587 | 8234 | 1.2145 | 32 |
| 29 | 7129 | 1.4028 | 7395 | 1.3522 | 7669 | 1.3040 | 7950 | 1.2579 | 8238 | 1.2138 | 31 |
| 30 | 7133 | 1.4019 | 7400 | 1.3514 | 7673 | 1.3032 | 7954 | 1.2572 | 8243 | 1.2131 | 30 |
| 31 | 7137 | 1.4011 | 7404 | 1.3506 | 7678 | 1.3024 | 7959 | 1.2564 | 8248 | 1.2124 | 29 |
| 32 | 7142 | 1.4002 | 7409 | 1.3498 | 7683 | 1.3017 | 7964 | 1.2557 | 8253 | 1.2117 | 28 |
| 33 | 7146 | 1.3994 | 7413 | 1.3490 | 7687 | 1.3009 | 7969 | 1.2549 | 8258 | 1.2109 | 27 |
| 34 | 7151 | 1.3985 | 7418 | 1.3481 | 7692 | 1.3001 | 7973 | 1.2542 | 8263 | 1.2102 | 26 |
| 35 | 7155 | 1.3976 | 7422 | 1.3473 | 7696 | 1.2993 | 7978 | 1.2534 | 8268 | 1.2095 | 25 |
| 36 | 7159 | 1.3968 | 7427 | 1.3465 | 7701 | 1.2985 | 7983 | 1.2527 | 8273 | 1.2088 | 24 |
| 37 | 7164 | 1.3959 | 7431 | 1.3457 | 7706 | 1.2977 | 7988 | 1.2519 | 8278 | 1.2081 | 23 |
| 38 | 7168 | 1.3951 | 7436 | 1.3449 | 7710 | 1.2970 | 7992 | 1.2512 | 8283 | 1.2074 | 22 |
| 39 | 7173 | 1.3942 | 7440 | 1.3440 | 7715 | 1.2962 | 7997 | 1.2504 | 8287 | 1.2066 | 21 |
| 40 | 7177 | 1.3934 | 7445 | 1.3432 | 7720 | 1.2954 | 8002 | 1.2497 | 8292 | 1.2059 | 20 |
| 41 | 7181 | 1.3925 | 7449 | 1.3424 | 7724 | 1.2946 | 8007 | 1.2489 | 8297 | 1.2052 | 19 |
| 42 | 7186 | 1.3916 | 7454 | 1.3416 | 7729 | 1.2938 | 8012 | 1.2482 | 8302 | 1.2045 | 18 |
| 43 | 7190 | 1.3908 | 7458 | 1.3408 | 7734 | 1.2931 | 8016 | 1.2475 | 8307 | 1.2038 | 17 |
| 44 | 7195 | 1.3899 | 7463 | 1.3400 | 7738 | 1.2923 | 8021 | 1.2467 | 8312 | 1.2031 | 16 |
| 45 | 7199 | 1.3891 | 7467 | 1.3392 | 7743 | 1.2915 | 8026 | 1.2460 | 8317 | 1.2024 | 15 |
| 46 | 7203 | 1.3882 | 7472 | 1.3384 | 7747 | 1.2907 | 8031 | 1.2452 | 8322 | 1.2017 | 14 |
| 47 | 7208 | 1.3874 | 7476 | 1.3375 | 7752 | 1.2900 | 8035 | 1.2445 | 8327 | 1.2009 | 13 |
| 48 | 7212 | 1.3865 | 7481 | 1.3367 | 7757 | 1.2892 | 8040 | 1.2437 | 8332 | 1.2002 | 12 |
| 49 | 7217 | 1.3857 | 7485 | 1.3359 | 7761 | 1.2884 | 8045 | 1.2430 | 8337 | 1.1995 | 11 |
| 50 | 7221 | 1.3848 | 7490 | 1.3351 | 7766 | 1.2876 | 8050 | 1.2423 | 8342 | 1.1988 | 10 |
| 51 | 7226 | 1.3840 | 7495 | 1.3343 | 7771 | 1.2869 | 8055 | 1.2415 | 8346 | 1.1981 | 9 |
| 52 | 7230 | 1.3831 | 7499 | 1.3335 | 7775 | 1.2861 | 8059 | 1.2408 | 8351 | 1.1974 | 8 |
| 53 | 7234 | 1.3823 | 7504 | 1.3327 | 7780 | 1.2853 | 8064 | 1.2401 | 8356 | 1.1967 | 7 |
| 54 | 7239 | 1.3814 | 7508 | 1.3319 | 7785 | 1.2846 | 8069 | 1.2393 | 8361 | 1.1960 | 6 |
| 55 | 7243 | 1.3806 | 7513 | 1.3311 | 7789 | 1.2838 | 8074 | 1.2386 | 8366 | 1.1953 | 5 |
| 56 | 7248 | 1.3798 | 7517 | 1.3303 | 7794 | 1.2830 | 8079 | 1.2378 | 8371 | 1.1946 | 4 |
| 57 | 7252 | 1.3789 | 7522 | 1.3295 | 7799 | 1.2822 | 8083 | 1.2371 | 8376 | 1.1939 | 3 |
| 58 | 7257 | 1.3781 | 7526 | 1.3287 | 7803 | 1.2815 | 8088 | 1.2364 | 8381 | 1.1932 | 2 |
| 59 | 7261 | 1.3772 | 7531 | 1.3278 | 7808 | 1.2807 | 8093 | 1.2356 | 8386 | 1.1925 | 1 |
| 60 | 7265 | 1.3764 | 7536 | 1.3270 | 7813 | 1.2799 | 8098 | 1.2349 | 8391 | 1.1918 | 0 |
| | cot | tan | cot | tan | cot | tan | cot | tan | cot | tan | |
| ′ | 54° | | 53° | | 52° | | 51° | | 50° | | ′ |

| ′ | 40° | | 41° | | 42° | | 43° | | 44° | | ′ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | tan | cot | tan | cot | tan | cot | tan | cot | tan | cot | |
| 0 | 8391 | 1.1918 | 8693 | 1.1504 | 9004 | 1.1106 | 9325 | 1.0724 | 9657 | 1.0355 | 60 |
| 1 | 8396 | 1.1910 | 8698 | 1.1497 | 9009 | 1.1100 | 9331 | 1.0717 | 9663 | 1.0349 | 59 |
| 2 | 8401 | 1.1903 | 8703 | 1.1490 | 9015 | 1.1093 | 9336 | 1.0711 | 9668 | 1.0343 | 58 |
| 3 | 8406 | 1.1896 | 8708 | 1.1483 | 9020 | 1.1087 | 9341 | 1.0705 | 9674 | 1.0337 | 57 |
| 4 | 8411 | 1.1889 | 8713 | 1.1477 | 9025 | 1.1080 | 9347 | 1.0699 | 9679 | 1.0331 | 56 |
| 5 | 8416 | 1.1882 | 8718 | 1.1470 | 9030 | 1.1074 | 9352 | 1.0692 | 9685 | 1.0325 | 55 |
| 6 | 8421 | 1.1875 | 8724 | 1.1463 | 9036 | 1.1067 | 9358 | 1.0686 | 9691 | 1.0319 | 54 |
| 7 | 8426 | 1.1868 | 8729 | 1.1456 | 9041 | 1.1061 | 9363 | 1.0680 | 9696 | 1.0313 | 53 |
| 8 | 8431 | 1.1861 | 8734 | 1.1450 | 9046 | 1.1054 | 9369 | 1.0674 | 9702 | 1.0307 | 52 |
| 9 | 8436 | 1.1854 | 8739 | 1.1443 | 9052 | 1.1048 | 9374 | 1.0668 | 9708 | 1.0301 | 51 |
| 10 | 8441 | 1.1847 | 8744 | 1.1436 | 9057 | 1.1041 | 9380 | 1.0661 | 9713 | 1.0295 | 50 |
| 11 | 8446 | 1.1840 | 8749 | 1.1430 | 9062 | 1.1035 | 9385 | 1.0655 | 9719 | 1.0289 | 49 |
| 12 | 8451 | 1.1833 | 8754 | 1.1423 | 9067 | 1.1028 | 9391 | 1.0649 | 9725 | 1.0283 | 48 |
| 13 | 8456 | 1.1826 | 8759 | 1.1416 | 9073 | 1.1022 | 9396 | 1.0643 | 9730 | 1.0277 | 47 |
| 14 | 8461 | 1.1819 | 8765 | 1.1410 | 9078 | 1.1016 | 9402 | 1.0637 | 9736 | 1.0271 | 46 |
| 15 | 8466 | 1.1812 | 8770 | 1.1403 | 9083 | 1.1009 | 9407 | 1.0630 | 9742 | 1.0265 | 45 |
| 16 | 8471 | 1.1806 | 8775 | 1.1396 | 9089 | 1.1003 | 9413 | 1.0624 | 9747 | 1.0259 | 44 |
| 17 | 8476 | 1.1799 | 8780 | 1.1389 | 9094 | 1.0996 | 9418 | 1.0618 | 9753 | 1.0253 | 43 |
| 18 | 8481 | 1.1792 | 8785 | 1.1383 | 9099 | 1.0990 | 9424 | 1.0612 | 9759 | 1.0247 | 42 |
| 19 | 8486 | 1.1785 | 8790 | 1.1376 | 9105 | 1.0983 | 9429 | 1.0606 | 9764 | 1.0241 | 41 |
| 20 | 8491 | 1.1778 | 8796 | 1.1369 | 9110 | 1.0977 | 9435 | 1.0599 | 9770 | 1.0235 | 40 |
| 21 | 8496 | 1.1771 | 8801 | 1.1363 | 9115 | 1.0971 | 9440 | 1.0593 | 9776 | 1.0230 | 39 |
| 22 | 8501 | 1.1764 | 8806 | 1.1356 | 9121 | 1.0964 | 9446 | 1.0587 | 9781 | 1.0224 | 38 |
| 23 | 8506 | 1.1757 | 8811 | 1.1349 | 9126 | 1.0958 | 9451 | 1.0581 | 9787 | 1.0218 | 37 |
| 24 | 8511 | 1.1750 | 8816 | 1.1343 | 9131 | 1.0951 | 9457 | 1.0575 | 9793 | 1.0212 | 36 |
| 25 | 8516 | 1.1743 | 8821 | 1.1336 | 9137 | 1.0945 | 9462 | 1.0569 | 9798 | 1.0206 | 35 |
| 26 | 8521 | 1.1736 | 8827 | 1.1329 | 9142 | 1.0939 | 9468 | 1.0562 | 9804 | 1.0200 | 34 |
| 27 | 8526 | 1.1729 | 8832 | 1.1323 | 9147 | 1.0932 | 9473 | 1.0556 | 9810 | 1.0194 | 33 |
| 28 | 8531 | 1.1722 | 8837 | 1.1316 | 9153 | 1.0926 | 9479 | 1.0550 | 9816 | 1.0188 | 32 |
| 29 | 8536 | 1.1715 | 8842 | 1.1310 | 9158 | 1.0919 | 9484 | 1.0544 | 9821 | 1.0182 | 31 |
| 30 | 8541 | 1.1708 | 8847 | 1.1303 | 9163 | 1.0913 | 9490 | 1.0538 | 9827 | 1.0176 | 30 |
| 31 | 8546 | 1.1702 | 8852 | 1.1296 | 9169 | 1.0907 | 9495 | 1.0532 | 9833 | 1.0170 | 29 |
| 32 | 8551 | 1.1695 | 8858 | 1.1290 | 9174 | 1.0900 | 9501 | 1.0526 | 9838 | 1.0164 | 28 |
| 33 | 8556 | 1.1688 | 8863 | 1.1283 | 9179 | 1.0894 | 9506 | 1.0519 | 9844 | 1.0158 | 27 |
| 34 | 8561 | 1.1681 | 8868 | 1.1276 | 9185 | 1.0888 | 9512 | 1.0513 | 9850 | 1.0152 | 26 |
| 35 | 8566 | 1.1674 | 8873 | 1.1270 | 9190 | 1.0881 | 9517 | 1.0507 | 9856 | 1.0147 | 25 |
| 36 | 8571 | 1.1667 | 8878 | 1.1263 | 9195 | 1.0875 | 9523 | 1.0501 | 9861 | 1.0141 | 24 |
| 37 | 8576 | 1.1660 | 8884 | 1.1257 | 9201 | 1.0869 | 9528 | 1.0495 | 9867 | 1.0135 | 23 |
| 38 | 8581 | 1.1653 | 8889 | 1.1250 | 9206 | 1.0862 | 9534 | 1.0489 | 9873 | 1.0129 | 22 |
| 39 | 8586 | 1.1647 | 8894 | 1.1243 | 9212 | 1.0856 | 9540 | 1.0483 | 9879 | 1.0123 | 21 |
| 40 | 8591 | 1.1640 | 8899 | 1.1237 | 9217 | 1.0850 | 9545 | 1.0477 | 9884 | 1.0117 | 20 |
| 41 | 8596 | 1.1633 | 8904 | 1.1230 | 9222 | 1.0843 | 9551 | 1.0470 | 9890 | 1.0111 | 19 |
| 42 | 8601 | 1.1626 | 8910 | 1.1224 | 9228 | 1.0837 | 9556 | 1.0464 | 9896 | 1.0105 | 18 |
| 43 | 8606 | 1.1619 | 8915 | 1.1217 | 9233 | 1.0831 | 9562 | 1.0458 | 9902 | 1.0099 | 17 |
| 44 | 8611 | 1.1612 | 8920 | 1.1211 | 9239 | 1.0824 | 9567 | 1.0452 | 9907 | 1.0094 | 16 |
| 45 | 8617 | 1.1606 | 8925 | 1.1204 | 9244 | 1.0818 | 9573 | 1.0446 | 9913 | 1.0088 | 15 |
| 46 | 8622 | 1.1599 | 8931 | 1.1197 | 9249 | 1.0812 | 9578 | 1.0440 | 9919 | 1.0082 | 14 |
| 47 | 8627 | 1.1592 | 8936 | 1.1191 | 9255 | 1.0805 | 9584 | 1.0434 | 9925 | 1.0076 | 13 |
| 48 | 8632 | 1.1585 | 8941 | 1.1184 | 9260 | 1.0799 | 9590 | 1.0428 | 9930 | 1.0070 | 12 |
| 49 | 8637 | 1.1578 | 8946 | 1.1178 | 9266 | 1.0793 | 9595 | 1.0422 | 9936 | 1.0064 | 11 |
| 50 | 8642 | 1.1571 | 8952 | 1.1171 | 9271 | 1.0786 | 9601 | 1.0416 | 9942 | 1.0058 | 10 |
| 51 | 8647 | 1.1565 | 8957 | 1.1165 | 9276 | 1.0780 | 9606 | 1.0410 | 9948 | 1.0052 | 9 |
| 52 | 8652 | 1.1558 | 8962 | 1.1158 | 9282 | 1.0774 | 9612 | 1.0404 | 9954 | 1.0047 | 8 |
| 53 | 8657 | 1.1551 | 8967 | 1.1152 | 9287 | 1.0768 | 9618 | 1.0398 | 9959 | 1.0041 | 7 |
| 54 | 8662 | 1.1544 | 8972 | 1.1145 | 9293 | 1.0761 | 9623 | 1.0392 | 9965 | 1.0035 | 6 |
| 55 | 8667 | 1.1538 | 8978 | 1.1139 | 9298 | 1.0755 | 9629 | 1.0385 | 9971 | 1.0029 | 5 |
| 56 | 8672 | 1.1531 | 8983 | 1.1132 | 9303 | 1.0749 | 9634 | 1.0379 | 9977 | 1.0023 | 4 |
| 57 | 8678 | 1.1524 | 8988 | 1.1126 | 9309 | 1.0742 | 9640 | 1.0373 | 9983 | 1.0017 | 3 |
| 58 | 8683 | 1.1517 | 8994 | 1.1119 | 9314 | 1.0736 | 9646 | 1.0367 | 9988 | 1.0012 | 2 |
| 59 | 8688 | 1.1510 | 8999 | 1.1113 | 9320 | 1.0730 | 9651 | 1.0361 | 9994 | 1.0006 | 1 |
| 60 | 8693 | 1.1504 | 9004 | 1.1106 | 9325 | 1.0724 | 9657 | 1.0355 | 1.000 | 1.0000 | 0 |
| | cot | tan | cot | tan | cot | tan | cot | tan | cot | tan | |
| ′ | 49° | | 48° | | 47° | | 46° | | 45° | | ′ |

# ANSWERS

## Ex. 1:

| | | |
|---|---|---|
| 1. $7\frac{1}{3}$ ft. | 2. 43′2″ | 3. 23,760 ft. per min. |
| 4. 184.5 sq. ft. | 5. 60 mi. per hr. | 6. 10 gal. |
| 7. \$3.36 | 8. 20 ft./sec. | 9. 69.4 sq. ft. |
| 10. .134 cu. ft.; 4.2 cu. ft. | 11. 8.32 lb. | 12. 12,480 lb. |
| 13. 40 cu. ft. | 14. 84,000 lb. | 15. 600 cu. ft./min. |

## Ex. 2:

| | | | | |
|---|---|---|---|---|
| 1. 8.75 oz. | 2. 1.8 in. | 3. 37.5 cm. | 4. 5.4 kg. | 5. 20″ |
| 6. 165 lb. | 7. 162.5 mm. | 8. 226.8 gm. | 9. 77.4 kg. | 10. 5′11.2″ |
| 11. 1900 cc. | 12. 10.6 qt. | 13. 1.59 pt. | 14. 7.6 l. | 15. 1300 sq. cm. |

## Ex. 3:

| | |
|---|---|
| 1. $\frac{24}{64}''$; $\frac{36}{16}''$; $\frac{52}{32}''$; $\frac{652}{64}''$ | 2. $4\frac{1}{2}$ eighths; $\frac{1}{16}''$ more; $\frac{3}{16}''$ less |
| 3. $\frac{40}{16}$; $\frac{26}{16}$; $\frac{51}{16}$; $\frac{12}{16}$ | 4. 23 |
| 5. $\frac{58}{16}$; $\frac{116}{32}$ | 6. No; yes; $\frac{3}{4}$, $\frac{7}{8}$, $2\frac{3}{64}$, $2\frac{1}{16}$ |

## Ex. 4:

| | | | | |
|---|---|---|---|---|
| 1. $1\frac{5}{12}$ | 2. $1\frac{3}{8}$ | 3. $1\frac{3}{4}$ | 4. $\frac{15}{16}$ | 5. $8\frac{3}{8}$ |
| 6. $6\frac{15}{16}$ | 7. $8\frac{5}{16}$ | 8. $8\frac{17}{32}$ | 9. $2\frac{3}{8}$ | 10. $2\frac{3}{16}$ |
| 11. $\frac{7}{64}$ | 12. $1\frac{15}{32}$ | 13. $5\frac{29}{32}''$ | 14. $29\frac{5}{8}''$ | 15. $80\frac{1}{4}''$ |
| 16. $1\frac{5}{64}''$ | 17. $1\frac{1}{8}''$; $2\frac{1}{16}''$; $\frac{11}{16}''$ | 18. $5\frac{5}{8}''$; $\frac{5}{8}''$ | 19. $17\frac{1}{16}''$ | 20. $10\frac{5}{16}''$ |

## Ex. 5:

| | | | | | |
|---|---|---|---|---|---|
| 1. $3\frac{7}{16}$ | 2. 87 | 3. 78 | 4. $5\frac{1}{2}$ | 5. 8 | 6. $\frac{5}{8}$ |
| 7. 160.3 gal. | 8. $5\frac{11}{16}''$ | 9. $2\frac{3}{4}''$ | 10. 19 | 11. $282\frac{3}{4}$ lb. | 12. 55 |

## Ex. 6:

| | | | | |
|---|---|---|---|---|
| 1. 0.276 | 2. 0.0154 | 3. 0.0007 | 4. 0.00425 | 5. 0.100 |

6. Three hundred seventy-nine and two-tenth thousandths.
7. Six-tenth thousandths.
8. Two hundred and two-tenth thousandths.
9. Seventy and one-half thousandths.
10. Two hundred eighty-one and six-tenth thousandths.
11. Ninety-six thousandths.
12. Four hundred forty-four and four-tenth thousandths.
13. Fifteen and eight-tenth thousandths.
14. One-quarter thousandth.

## Ex. 7:

| | | | |
|---|---|---|---|
| 1. 2.5175″ | 2. 2.3758″ | 3. 1.192″ | 4. .6175″ |
| 5. .694″ | 6. 4.3917″ | 7. 1.372″ | 8. 4.214″; 1.81″ |
| 9. 1.503″; 3.678″ | 10. .0007″ | 11. .239″ | 12. 1.444″; 2.786″; .7883″ |

**Ex. 8:**

1. 5.95″ 2. 445.09 lb. 3. .682″ 4. 216.5 lb. 5. .108″
6. 7.57 lb. 7. .648″ 8. 139.7 cal. 9. $336.86 10. .10752″

**Ex. 9:**

1. 73.5 cu. ft. 2. 30.5 cu. ft. 3. 16.61 cm. 4. 36 5. 5.438″ 6. 96

**Ex. 10:**

1. .125 2. .09375 3. .4375 4. .234375 5. .28125
6. .9375 7. .84375 8. .6875 9. 3.142857 10. .555556
11. .9230769 12. .181818 13. .171875 14. .765625 15. .208333
16. .19531 17. $\frac{7}{16}$ 18. $\frac{13}{16}$ 19. $\frac{1}{16}$ 20. $\frac{1}{320}$
21. $\frac{13}{32}$ 22. $\frac{29}{32}$ 23. $\frac{5}{64}$ 24. $\frac{59}{64}$ 25. $\frac{17}{32}$

**Ex. 11:**

1. (*a*) $\frac{7}{32}$ (*b*) $\frac{27}{32}$ (*c*) $\frac{3}{32}$ 2. (*a*) $\frac{31}{64}$ (*b*) $\frac{11}{64}$ (*c*) $\frac{51}{64}$
3. 1.3866 to 1.3946; 6.246 to 6.254; .961 to .969; 1.038 to 1.046
4. .88125 to .93125; 1.13125 to 1.18125; 1.1625 to 1.2125; .5375 to .5875; 2.100 to 2.150

**Ex. 12:**

1. .908″; .035″ 2. $x=2\frac{5}{8}''$, $y=2\frac{1}{4}''$; $x=2.640''$, $y=2.2575''$
3. $\frac{13}{16}''$, $2\frac{11}{16}''$, $3\frac{5}{32}''$, $1\frac{3}{16}''$ 4. .762″; 2.637″; 3.258″; 1.141″
5. At least .003″, but not more than .007″

**Ex. 13:**

1. (*a*) .465 (*b*) .904 (*c*) .595 (*d*) .132 (*e*) .022 (*f*) .991
2. (*a*) .3473 (*b*) .8839 (*c*) .4662 (*d*) .2114 (*e*) .5776 (*f*) .6241

**Ex. 14:**

1. .1004″, .144″, .150″ 2. .1002″, .350″, .103″
3. .1005″, .200″, .126″ 4. .550″, .116″
5. .1002″, .150″; .108″, .040″ 6. .1008″, .200″, .133″
7. 3.000″, .1001″, .700″, .106″ 8. 2.000″, .1002″, .500″, .107″
9. 1.000″, .1009″, .650″, .103″
10. 4.000″, 3.000″, 1.000″, .900″, .500″, .140″, .1002″
11. 4.000″, 3.000″, .144″, .050″, .1004″ 12. 3.000″, .1008″, .800″, .109″

**Ex. 15:**

1. 57.22 lb. 2. 94.6 H.P. 3. 2769 r.p.m.
4. 90 H.P. 5. 7½ lb. 6. $12.04; $8.74
7. 21 lb.; 40½ lb.; 88½ lb. 8. 27.5 lb. 9. 6 pieces
10. 4.728″ and 4.872″ 11. 217½ lb.; 797½ lb.; 435 lb.

**Ex. 16:**

1. 16⅔% 2. 15% 3. 12% 4. 25.7%
5. 4.5% 6. 12½% 7. 40% 8. 1.6%
9. .0332% 10. 25% 11. 1.8% 12. 8.6%

**Ex. 17:**

*1.* 156; 195 *2.* 22.5″ *3.* 1000 ft. *4.* 88 lb. *5.* $160,000 *6.* $48.50

**Ex. 18:**

*1.* $10.80 *2.* $1.60 *3.* 20% *4.* $70 *5.* 12½%
*6.* $22,400 *7.* 3.6 oz. *8.* 6⅔% *9.* 390 cu. ft. *10.* 3⅛%
*11.* 368 *12.* 4.2% *13.* $560 *14.* $115.63 *15.* $120

**Ex. 19:**

*1.* 2:3 *2.* 1:6 *3.* 2:3
*4.* 8:19 *5.* 13:17 *6.* 17½″
*7.* 15:23 *8.* 7 in. *9.* .04 cm.
*10.* 7 in.

**Ex. 20:**

*1.* 5:2; 2:5 *2.* 943:1000 *3.* 17½″, 24½″ *4.* 30°, 60°, 90°
*5.* 19 liters *6.* 17.32″; 3″ *7.* 1:4; 55 *8.* 200.26 cm.
*9.* 18 lb. *10.* 300, 700, 1000 *11.* .88 *12.* 460 lb.
*13.* 1:4 *14.* 2:5; 5:3; 30% *15.* 5:8; 2:5; 33⅓%

**Ex. 21:**

*1.* 1″×2⅛″ *2.* 2½″×1¾″
*3.* 8″; 20″; 15′; 6½″ *4.* 2½′; 1″; ¼″; 1.2″; 40′; 72½′
*5.* (*a*) 1¾″; 65′ (*b*) 3″; 60″ (*c*) 1½″; 19′ (*d*) 12½″; 11½ mi.
(*e*) 1⅛″; 180 ft.
*6.* 14″×24½″ *7.* 3¼″×4½″ *8.* 8.17″ *9.* 14′×23′ *10.* 27′×37½′
*11.* 37½ ft. *12.* 2½″
*13.* 1″=1′; 1″=10′; 3″; 12′; 10″; 1″=2′ *14.* 3½′×7′
*15.* L.R., 12′×20′; Dinette, 9½′×10′; B.R., 11½′×17′; Alcove, 8½′×12½′

**Ex. 22:**

*1.* 20 *2.* 40 *3.* 21 *4.* 20 *5.* 12 *6.* 27
*7.* 9 *8.* 42 *9.* 6 *10.* 24 *11.* 6 *12.* 28

**Ex. 23:**

*1.* 1.8″ *2.* $56 *3.* 525 pieces *4.* 2380 mi.
*5.* 525 ohms *6.* 780 lb. *7.* 42 ft.; 250 lb. *8.* 326⅔; 642.9
*9.* 288 cu. ft.; 125 lb. *10.* .56 mi. nearer

**Ex. 24:**

*1.* $C_2=12$; $R_2=60$ *2.* 576 cu. in.; 72 lb. per sq. in. *3.* $w_2=3$ lb.; $d_2=8''$
*4.* .0024 *5.* 24 r.p.m. *6.* 24″
*7.* 700 r.p.m. *8.* 13 teeth *9.* 2500 r.p.m.
*10.* 27″

**Ex. 25:**

1. Twice one number increased by three times another number.
2. Five times one number diminished by half another number.
3. One number decreased by a second and increased by a third, and the result divided by three.
4. Two-thirds of a number increased by eight.
5. One number decreased by the reciprocal of another number.
6. Five times the product of two numbers.
7. The difference between two numbers divided by twice a third number.
8. Nine-fifths of a number increased by thirty-two.
9. Twice the product of the length by the width added to twice the product of the length and height, added to twice the product of the width and height.
10. The difference of two numbers divided by the difference of two other numbers.

**Ex. 26:**

1. 11.4
2. 10.33
3. 90
4. 3.586
5. 179.7
6. .0689
7. 211.1
8. 2, 3, 4, 5, 7½
9. 0, 8, 16, 24, 32
10. 3, 6, 7½, 9, 18
11. 0, ¼, ½, ¾, 1¼
12. 2⅔, 5⅓, 8, 10⅔, 16
13. ⅔, 2, 3⅓, 4⅔, 6⅔
14. 0, .32, .64, 3.19
15. 0, 4½, 9, 18, 45

**Ex. 27:**

1. $+6$; $+38$; $0$; $-9$; $-22$; $-12$; $+12$
2. $+23$; $+13$; $-6$; $0$; $-2$; $-36$; $0$
3. $-63$; $+60$; $-60$; $+2$; $+6.2$; $-72k$; $+30m$; $-12a$; $-48r$
4. $-9$; $-9$; $+3$; $+12$; $-2$; $+1$; $-2$; $-8/a$; $-8x$

**Ex. 28:**

1. $21p$; $26r$; $9m$; $8.6h$; $-22\frac{1}{2}x$
2. $4k$; $-6a$; $26xy$; $-36lw$; $+2\frac{1}{2}ab$
3. $34x$
4. $26k$
5. $4h+4l+4w$
6. $12e$
7. $8\frac{1}{4}k$
8. $9d$

**Ex. 29:**

1. $32m$; $30k$; $15pq$; $12abh$; $12a^3b$; $66h$; $\frac{1}{3}st$; $5r^2h$; $\frac{4}{3}R^3$; $7.2a^2t^3$; $12a^3b^3$; $5x^7$; $m^5$; $6x^2y^2$; $\frac{1}{6}a^5$
2. $7m$; $8$; $24$; $3/x$; $7\frac{1}{2}k$; $7y$; $12a^2/b$; $5$; $3r/s^3$; $-8d^2/c$; $2Rh$; $-2a^2b^2$

**Ex. 30:**

1. $2l+2w$
2. $ka+kb-kc$
3. $a+nd-d$
4. $IR+Ir$
5. $P+PRT$
6. $S=\frac{na}{2}+\frac{nl}{2}$
7. $\frac{hB}{2}+\frac{hb}{2}$
8. $\pi R^2-\pi r^2$
9. $Lt_0-Lt_1$
10. $2\pi rl+2\pi r^2$
11. $na+\frac{n^2d}{2}-\frac{nd}{2}$
12. $V=\frac{hb_1}{6}+\frac{hb_2}{6}+\frac{2}{3}hM$
13. $p\,(1+i)$
14. $h\,(p-q)$
15. $I^2\,(R+r)$
16. $2a\,(b-c)$
17. $P\,(1+rt)$
18. $\frac{1}{2}h\,(B_1+B_2)$
19. $\pi D\,(l+1)$
20. $2\pi R\,(l+R)$
21. 100; 20
22. 21.58
23. .5625
24. 8448

**Ex. 31:**

| | | | |
|---|---|---|---|
| *1.* 64 | *5.* 1 | *9.* ¼ | *13.* 27/64 |
| *2.* 256 | *6.* 27 | *10.* 4/9 | *14.* .0001 |
| *3.* 216 | *7.* 400 | *11.* 1/64 | *15.* .000001 |
| *4.* 1728 | *8.* 343 | *12.* 1/100 | *16.* .000001 |

**Ex. 32:**

| | | |
|---|---|---|
| *1.* 3,400,000 | *4.* 430,000,000,000 | *7.* 862,000,000 |
| *2.* 6,200,000,000 | *5.* 5,200,000 | *8.* 24,630,000,000,000 |
| *3.* 91,000,000,000 | *6.* 1,380,000,000,000 | *9.* .000035 |
| *10.* .00000023 | *13.* $37\times10^6$ | *16.* $4.9\times10^9$ |
| *11.* .0000000049 | *14.* $5.8\times10^6$ | *17.* $4\times10^{-5}$ |
| *12.* .000000000326 | *15.* $12.4\times10^6$ | *18.* $2.8\times10^{-7}$ |
| *19.* $3.92\times10^{-4}$ | *20.* $7.6\times10^{-10}$ | *21.* 5,882,500,000,000 |
| *22.* $58.93\times10^{-5}$ | | |

**Ex. 33:**

*1.* $\sqrt[3]{a}$; $\sqrt[4]{x}$; $\sqrt{5a}$; $\sqrt[3]{p^2}$; $\sqrt{A^3}$; $\sqrt{mn}$

*2.* $p^{1/2}$; $k^{1/4}$; $a^{2/3}$; $(10x)^{1/5}$; $\left(\frac{2s}{g}\right)^{1/2}$; $\left(\frac{A}{\pi}\right)^{1/2}$

*3.* 12; 1½; 8; 4; $6a$; 12; 216; 4,000,000

*4.* 1/5; 8; 9; .2; 16; 2; .3; 8; ½; .01; $\frac{1}{25x^2}$; $\frac{1}{4a^2}$

**Ex. 34:**

| | | | |
|---|---|---|---|
| *1.* 364 in. | *2.* 37½ in. | *3.* 456 | *4.* 64.4° |
| *5.* $510 | *6.* 2.45 | *7.* 200,000 | *8.* 15.84 |
| *9.* 59,040 | *10.* 26.4 cal. | *11.* 153.6 H.P. | *12.* 128,060 sq. ft. |

**Ex. 35:**

| | | | | | |
|---|---|---|---|---|---|
| *1.* 13 | *2.* 32 | *3.* .35 | *4.* 20 | *5.* 12 | *6.* 15 |
| *7.* 14 | *8.* 320 | *9.* 12 | *10.* 36 | *11.* 16 | *12.* 160 |

**Ex. 36:**

| | | | |
|---|---|---|---|
| *1.* $l=\frac{A}{w}$ | *2.* $M=DV$ | *3.* $T=\frac{D}{R}$ | *4.* $R=\frac{E}{I}$ |
| *5.* $R=\frac{C}{2\pi}$ | *6.* $s=\frac{d}{1.4}$ | *7.* $P=\frac{kT}{V}$ | *8.* $T=\frac{I}{PR}$ |
| *9.* $h=\frac{2A}{b}$ | *10.* $r=\frac{S}{2\pi h}$ | *11.* $w_2=\frac{l_1w_1}{l_2}$ | *12.* $t^2=\frac{2S}{g}$ |

**Ex. 37:**

| | | | | | |
|---|---|---|---|---|---|
| *1.* 22 | *2.* 14 | *3.* 38 | *4.* 95 | *5.* 14 | *6.* 36 |
| *7.* 3½ | *8.* 10.7 | *9.* 2.1 | *10.* 9.25 | *11.* 22.5 | *12.* 38.3 |

**Ex. 38:**

1. 2
2. 4
3. $\frac{n-p}{k}$
4. $\frac{C-q}{4}$
5. $\frac{P-2l}{2}$
6. $\frac{V-E}{r}$
7. $\frac{12D-TL}{12}$
8. $\frac{V-v_0}{g}$
9. $\frac{2A-bh}{h}$
10. $\frac{A-P}{PT}$
11. $\frac{l-a}{n-1}$
12. $N-2PC$
13. $\frac{DL}{2\pi}$
14. $NW+W; \frac{s}{N+1}$
15. $\frac{FRg}{v^2}; \frac{Wv^2}{gF}$
16. $\frac{Q(t_1-t_0)+wh}{w}$

**Ex. 39:**

1. 4.2
2. 5.5
3. 3.5
4. 6.4
5. 7.7
6. 4.6
7. 7.4
8. 8.6
9. 11.4
10. 12.2
11. 13.1
12. 14.1
13. 9.3
14. 7.4
15. 2.5
16. 19.5
17. 23.8
18. 54.2
19. 8.3
20. 20.7

**Ex. 40:**

1. 7.4
2. 7.6
3. 13.3
4. 32.0
5. 6.4
6. 11.22
7. 166.7
8. 35.4 mi.
9. 40 sq. ft.
10. 9.2 in.

**Ex. 41:**

1. $v=\sqrt{\frac{2E}{m}}$
2. $t=\sqrt{\frac{2v}{g}}$
3. $h=\frac{2m^2}{3}$
4. $h=\frac{s}{2}\sqrt{3}$
5. $s=\sqrt{\frac{4A}{\sqrt{3}}}$
6. $R=\sqrt{\frac{3V}{\pi h}}$
7. $C=\frac{1}{4\pi^2 f^2 L}$
8. $C=\sqrt{\frac{6I}{\pi ab}}$
9. $v=10\sqrt{\frac{11gP}{W}}$
10. $R=\frac{1}{C}\sqrt{E^2-(2\pi nLC)^2}$
11. $h=\frac{.36P^2}{A^2}$
12. 38.0 sq. ft.

**Ex. 42:**

1. 2.7; —1.4
2. 12.4; —2.4
3. 3.3; —4.3
4. 1.5; —1.3
5. 3.8; —.3
6. 3.0; —5.0

**Ex. 43:**

1. Doubled; halved; divided by 10; multiplied by 1½; divided by 8
2. Multiplied by 9; divided by 6; constant; multiplied by 4
3. $V$ decreases; $P$ decreases; $P$ is doubled; $V$ is divided by 3; $P$ is divided by 5
4. Multiplied by 4; divided by 9; multiplied by 25; divided by 100

**Ex. 48:**

*1.* 2.9694 *2.* 2.6112 *3.* 9.7024—10 *4.* 1.8554
*5.* 6.7210—10 *6.* 5.8082 *7.* .9051 *8.* 7.5328—10
*9.* 9.8408—10 *10.* 6.6532—10 *11.* 3.6817 *12.* 2.3746

**Ex. 49:**

*1.* 24.02 *2.* 4.397 *3.* 265.3 *4.* .4116 *5.* 25.06 *6.* .1996
*7.* 5.788 *8.* .02619 *9.* 276.4 *10.* 52,663 *11.* .008134 *12.* 1112

**Ex. 50:**

*1.* 1.333 *2.* .00986 *3.* 1051 *4.* 4185 *5.* .2374 *6.* .004759

**Ex. 51:**

*1.* 9.110 *2.* 3.894 *3.* .05604 *4.* 2.116 *5.* 15.78
*6.* 2.222 *7.* 1.486 *8.* 65,000 *9.* .01985 *10.* 19,620
*11.* 465.8 *12.* 10.72 *13.* 2.466 *14.* 4.444 *15.* 5.46 in.

**Ex. 52:**

*1.* 1.796 *2.* 13.75 *3.* 1.181 *4.* 4.166 *5.* 2.56
*6.* 1.14 *7.* 1.79 *8.* 47.9 *9.* 18.3 yr. *10.* $678.70; 4.3 yr.

**Ex. 53:**
*1.* 950 *2.* 565 *3.* 57.4 *4.* .231 *5.* 123.5 *6.* 116.1 *7.* 710
*8.* 4.03 *9.* 38.3 *10.* 4.16 *11.* 79.2 *12.* 1.44 *13.* 5.21 *14.* 5.12

**Ex. 54:**

*1.* 6.63 *2.* 4.98 *3.* 8.4 *4.* 11.7 *5.* 2.87 *6.* .92
*7.* .0179 *8.* .253 *9.* 58.7 *10.* .07 *11.* 1.75 *12.* .087

**Ex. 55:**

*1.* 51°; 67½°; 45°40′; 22°18′48″
*2.* 148°; 135°; 90°; 58°; 122°31′; 19°15′12″
*3.* 85°20′ *4.* 100° *5.* 85° *6.* 56°; 34° *7.* 34° *8.* 50°

**Ex. 56:**

*1.* 142°; 68°; 25° *2.* $x=14°$; $y=166°$; $z=76°$
*3.* 120°; 58°; 360° *4.* 115°; 68°; 130°; 68°; 132°
*5.* 64°45′ *6.* 133°
*7.* AM, MC, AB; BC, BM, BM *8.* OR, point O, OS; OM, OB, OP

**Ex. 57:**

*1.* 75° 24′ *2.* 16° *3.* 45° 27′ *4.* 28° 35′ 50″ *5.* 64° 38′ *6.* 109° 12′

**Ex. 58:**

*1.* 12.04 in. *2.* 20.6 ft. *3.* 11.8 in. *4.* 6.4 in. *5.* 22.4 in.
*6.* 8.9 in. *7.* 7.2 in.; 10.8 in. *8.* 22′ 9″ *9.* 5.66 in. *10.* 9.798 in.

**Ex. 59:**

*1.* 200° *2.* 215° *3.* 158° *5.* $\widehat{AB}=40°$; $\widehat{DC}=40°$; $\widehat{BC}=220°$
*6.* $\angle P=95°$; $\angle Q=65°$; $\angle R=85°$: $\angle S=115°$

**Ex. 60:**

*1.* 15.5 in. *2.* 17° 10′; $a=\frac{\beta}{2}$ *3.* 9.899 in.
*4.* 4.58 in.; $x=\frac{1}{2}d\sqrt{2}$ *5.* 220 ft. per sec. *6.* 160 r.p.m.
*7.* 9.05 in. *8.* $\phi=57°\ 30'$; $\phi=2a$ *9.* 14.14″; 44.44″
*10.* 288 times *11.* 1.225″; $y=s\sqrt{2}$ *12.* Twice as long
*13.* 5.66″ *14.* 54° *15.* 6″
*16.* 1½ revolutions *17.* $k=\frac{D}{2}\sqrt{3}$; 2.70″ *18.* 50.3 ft.
*19.* 1.06″ *20.* 2263 ft. per min. *21.* 3 in.
*22.* 13.13 in. *23.* 17.34 in. *24.* 23.6 in.
*25.* 110 ft.

**Ex. 64:**

*1.* 42.4 sq. in. *2.* 22.2 sq. in. *3.* 61.2 sq. in.
*4.* 288 sq. in. *5.* 2 sq. in. *6.* 396 sq. ft.
*7.* 36 sq. in. *8.* 26.5 sq. in. *9.* 896 sq. ft.
*10.* 176 sq. in. *11.* 2400 sq. rd.; 15A. *12.* 320 sq. ft.
*13.* 56 sq. in. *14.* 17.8 sq. in. *15.* 2.59 sq. in.

**Ex. 65:**

*1.* 3.14 sq. in. *2.* 10.63 in. *3.* 366.7 sq. in.
*4.* 245.6 sq. in. *5.* 2:3; 4:9 *6.* 17.3%
*7.* 6.48 sq. in. *8.* 22.2 sq. in. *9.* 36.5%
*10.* 7.2 sq. in. *11.* 3.4 sq. in. *12.* 10.1 sq. in.
*13.* 484 sq. in. *14.* 50.3 sq. in. *15.* 21,384 lb.
*16.* 3 in. *17.* 40% *18.* 10 in.
*19.* 12.57 sq. in.; 11.78 sq. in. *20.* 209.3 sq. in.

**Ex. 66:**

*1.* 72 lb. *2.* 136 sq. ft. *3.* 46.1 cu. ft.
*4.* 6.29 cu. in.; 1.95 lb. *5.* 24.75 cu. in. *6.* $27.20
*7.* 38.29 lb. *8.* 450 sq. ft. *9.* 12½ lb.
*10.* 37.4 gal. *11.* 125 *12.* 11.7 cu. in.
*13.* 28.9 cu. in. *14.* 156.8 cu. in. *15.* 2282.8 ft.
*16.* 6.9 cu. yd. *17.* 12 in. *18.* 5,715 lb.
*19.* 6.48 in. *20.* 6,594 sq. ft. *21.* 29.2 lb.
*22.* 33,880 cu. yd.

**Ex. 67:**

*1.* 4.8 gal. *2.* 14.5 cu. yd. *3.* 3,014 cu. cm. *4.* 33⅓%
*5.* 847.8 sq. in. *6.* 242.5 cu. in. *7.* 187.7 cu. ft. *8.* 44.9 cu. yd.

9. 990 sq. in. 10. 7.07 cu. ft. 11. 87.5 sq. cm. 12. 268.7 cu. in.
13. 200.8 cu. in. 14. 1:4 15. 122,033 gal. 16. 3.6 cu. in.

**Ex. 68:**

1. 80 in. 2. 108 sq. ft. 3. BD=8.94; AB=12.00
4. AC=15; BC=20 5. 135 ft. 6. (a) 3.2″, 12.8″; (b) 16″
7. (a) 7.2″; (b) 9.6″; 5.4″ 8. 12″ and 15″, respectively

**Ex. 69:**

1. Half; one-fourth 2. $16\pi$, or 50.24″
3. Area, 4 times; volume, 8 times 4. 3:4; 9:16
5. 28.28 in.
6. Circumference, 10% less; area, 19% less
7. 10 in. 8. 9:16 9. 125% 10. 5.19 to 1
11. L.A.=401.9 sq. in.; V=927.9 cu. in. 12. 56.25% increase
13. T.A.=195.7 sq. in.; V=169.4 cu. in. 14. 12 sq. in.
15. 72.8% 16. 125%
17. Twice as large 18. 400 times larger
19. 4 times; 4 times 20. 125%

**Ex. 70:**

1. 26.3 in. 2. 18° 26′ 3. 12.16; 80° 32′; 18° 58′
4. 176.3 ft. 5. 4.83 6. 36° 52′

**Ex. 71:**

1. 24.8 in. 2. 40.0″; 24° 43′; 130° 34′ 3. 21.47″; 67° 58′
4. 3.39-inch square 5. 71° 47′; 36° 26′; 15.2 in. 6. 7.46″; 7.73″
7. 10.77″; 81° 18′ 8. 3.696″ 9. 4.33″
10. 12.8″; 15.8″ 11. 10.77′; 27.58′ 12. 59° 30′; 1.89″
13. .343″ 14. 1.805″ 15. 92° 48′

**Ex. 72:**

1. 167.7 ft. per min. 2. 212.13 mi. per hr. N.; same for E.
3. 79.5 lb.; 127.2 lb. 4. 459.6 lb.; 385.7 lb.
5. 456.75 lb.; 203.35 lb. 6. 45.3 ft. per sec.; 21.3 ft. per sec.
7. 1000 lb.; 36° 52′ with the horizontal 8. 176.78 lb.

**Ex. 73:**

1. 5.358″ 2. 10.39″ 3. 4.50″ 4. 20.5″

**Ex. 74:**

1. 5° 12′ 2. 26° 59′ 3. 33°; .5924″
4. 33° 40′ 5. 2.5584″ 6. 25° 22′; 12° 50′

**Ex. 75:**

1. 6.96″ 2. 3.33″ 3. 5.5″ 4. 1.46″
5. 4.402″ 6. 2.55″; 3.56″ 7. .048″ 8. 39° 58′
9. 1.06″ 10. .96″ 11. 73° 44′ 12. .65″

**Ex. 76:**

*1.* $a=13.6''$; $b=16.4''$; $C=82°\ 50'$
*2.* $a=31.4''$; $c=46.1''$; $B=28°\ 50'$
*3.* $b=25.7''$; $c=28.5''$; $C=79°\ 3'$
*4.* $b=11.79''$; $A=94°\ 5'$; $C=38°\ 35'$
*5.* $c=29.72''$; $A=28°\ 25'$; $B=16°\ 35'$

**Ex. 77:**

*1.* $c=30.4''$; $C=82°\ 32'$; $B=31°\ 28'$
*2.* $A=31°\ 35'$; $B=38°\ 57'$; $C=109°\ 28'$
*3.* $a=4.3''$; $A=46°\ 15'$; $C=23°\ 45'$
*4.* $A=55°\ 17'$; $B=39°\ 37'$; $C=85°\ 6'$
*5.* $b=21.5''$; $B=96°\ 17'$; $C=56°\ 13'$

**Ex. 78:**

*1.* 115.97 sq. in.
*2.* 430.98 sq. in.
*3.* 1175.6 sq. cm.
*4.* 4.26 in.
*5.* 403.1 sq. in.
*6.* $x=17°\ 21'$; $y=107°\ 21'$
*7.* 5.75 in.
*8.* 40.9 ft.
*9.* 1.53 in.
*10.* 82° 6′; 97° 54′

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